Instructions:
Do two E problems and two problems in the area C or M in which you
signed up.
Write your letter code on all of your answer sheets.
If you think that a problem has been stated incorrectly, mention this to
the proctor and indicate your interpretation in your solution. In such cases,
do not interpret the problem in such a way that it becomes trivial.

E1. Let $L$ be a language which includes a unary relation symbol $R$. Let $\phi$
be an $L$-sentence and $\Gamma$ a set of $L$-sentences neither of which contains the
symbol $R$. If $\Gamma$ proves $\phi$ in the language $L$, must there be a deduction of $\phi$
from $\Gamma$ in which $R$ does not occur (i.e., in the language $L - \{R\}$)? If so,
prove that there is always such a deduction; and if not, describe $\Gamma$ and $\phi$
which provide a counterexample.

E2. Show that there exists an $N \models PA$ and an $a \in N \setminus N$ so that $a$
is definable in $N$.

E3. Let $\alpha$, $\beta$ and $\gamma$ be ordinals. Prove that the six sums,
$$\alpha + \beta + \gamma, \quad \alpha + \gamma + \beta,$$
$$\beta + \alpha + \gamma, \quad \beta + \gamma + \alpha,$$
$$\gamma + \alpha + \beta, \quad \gamma + \beta + \alpha,$$
cannot all be different.
Computability Theory

C1. Say that a computable function $f$ has a limit if for all $x$, $\lim_s f(x,s)$ exists. Show that the index set $\{ e \mid \phi_e \text{ has a limit} \}$ is $\Pi_3$-complete.

C2. Show that no 1-generic set computes a diagonally noncomputable function. (Recall that a function $f$ is diagonally noncomputable if for all $e$, $f(e) \neq \phi_e(e)$.)

C3. Show that $a$ is a hyperimmune degree if and only if $a$ computes a function $f \leq_T A$ which agrees with every total computable function infinitely often. (Recall that a Turing degree $a$ is hyperimmune if it computes a function $g$ which is not dominated by any total computable function.)
Model Theory

M1. Let $T$ be a theory in the language of a single unary function $f$ stating that $f$ has no loops (i.e., for every $n > 0$ and every $x$, $f^n(x) \neq x$) and for every $x$, there are infinitely many $y$ with $f(y) = x$. Show that $T$ has quantifier elimination, is complete and not $\kappa$-categorical for any infinite cardinal $\kappa$.

M2. Find a complete theory $T$ in a countable first-order language such that the space $S_1(T)$ of 1-types is uncountable but $T$ is atomic. (Recall that $T$ is atomic if every formula $\phi(x_1, \ldots, x_n)$ is contained in a principal $n$-type.)

M3. Show that a complete countable first-order theory with infinite models is $\aleph_0$-categorical if and only if all of its models are pairwise back-and-forth equivalent.

Recall $A$ and $B$ are back-and-forth equivalent if there is a set $I$ comprised of pairs $(\bar{a}, \bar{b})$ where $\bar{a} \subset A$ and $\bar{b} \subset B$ such that the following hold:

- $(\emptyset, \emptyset) \in I$,
- If $(\bar{a}, \bar{b}) \in I$, then $|\bar{a}| = |\bar{b}| < \omega$ and $\text{tp}_{q.f.}^A(\bar{a}) = \text{tp}_{q.f.}^B(\bar{b})$ (i.e., their quantifier-free types coincide),
- If $(\bar{a}, \bar{b}) \in I$ and $c \in A$, then there exists a $d \in B$ so that $(\bar{a}c, \bar{b}d) \in I$,
- and
- If $(\bar{a}, \bar{b}) \in I$ and $d \in B$, then there exists a $c \in A$ so that $(\bar{ac}, \bar{bd}) \in I$. 
Sketchy Answers or Hints

E1 ans. Straightforward application of the Completeness theorem: If $\Gamma$ proves $\phi$, then any model $M$ of $\Gamma$ is a model of $\phi$. The same then also holds for any model $M$ of $\Gamma$ in the language $L - \{R\}$, so again by Completeness, there is a deduction of $\phi$ from $\Gamma$ in the language $L - \{R\}$.

E2 ans. By the Incompleteness Theorem, we can find a $\Delta_0$-sentence $\phi(x)$ such that $\mathbb{N} \models \forall x \neg \phi(x)$ but $\text{PA} + \exists x \phi(x)$ is consistent. Then any model $\mathcal{N} \models \text{PA} + \exists x \phi(x)$ contains, by induction, a least witness $a$ for $\phi$, which must be both nonstandard and definable.

E3 ans. Write $\alpha, \beta$ and $\gamma$ in Cantor normal form as

$$\omega^\alpha a_n + \cdots + \omega^\alpha a_0, \ \omega^\alpha b_n + \cdots + \omega^\alpha b_0, \ \omega^\alpha c_n + \cdots + \omega^\alpha c_0,$$

respectively, where $a_n, \ldots, a_0, b_n, \ldots, b_0, c_n, \ldots, c_0$ are non-negative integers. Now use the fact that for $\delta < \epsilon$, $\omega^\delta \cdot d + \omega^\epsilon = \omega^\epsilon$.

C1 ans.

C2 ans.

C3 ans.

M1 ans. Proof of QE 1: Let’s consider a formula of the form $\exists y (\phi(\bar{x}, y))$ where $\phi$ is a conjunction of literals: $\bigwedge \pm t_1(\bar{x}, y) = t_2(\bar{x}, y)$. Each term can take in only one parameter (as $f$ is unary), so this really is $\bigwedge \pm t_1(x_i) = t_2(y)$. Whether or not this configuration can hold is determined only by the configuration of $\bar{x}$ - this can be verified in cases: The only hard-ish case is when two $x$’s are connected and $f(x_1) = y$ and $f(y) = x_2$, but $f^2(x_1) \neq x_2$ Proof of QE 2 (the better one): We show that every type $p \in S_1(A)$ is determined by its q.f.-type. Suppose we had a model $M$ with
2 element realizing the q.f.-type $p$. If the q.f.-type says it’s connected to an $a \in A$, then show that the two elements are automorphic in $M$ over $A$. If it’s not connected, then in a saturated elementary extension (which must be homogeneously splitting), it’s easy to automorph the two elements while fixing $A$. Completeness follows from QE-ness. Not $\aleph_0$-categorical: one model with 1 tree and one model with 2 trees. Not $\aleph_1$-categorical: One model with $\aleph_1$-splittings on a single tree, and one model with $\aleph_0$-splittings but $\aleph_1$-many trees.

**M2 ans.** Take a tree in $2^{<\omega}$ with infinitely many paths but a dense set of isolated paths. Let $T$ be the theory associated to this tree (ie. the 1-types in $T$ are exactly the paths through this tree, and all 2-types are controlled by 1-types. This works

**M3 ans.** $\leftarrow$: Take any 2 countable models. The back-and-forth builds an isomorphism. $\rightarrow$: Using Ryll-Nardzewski, build the back-and-forth. Given $(\bar{a}, \bar{b}) \in I$ by stage $s$, and any element $c \in A$, let $\phi$ isolate the type of $c$ over $\bar{a}$. $\exists x \phi(x)$ is in the type of $\bar{a}$, thus also of $\bar{b}$. Let $d$ be a realization of this formula, and put $(\bar{a}c, \bar{b}d)$ into $I$ at stage $s + 1$. Do the back direction too.