1. (Jan-99.4) Let $V$ be finite-dimensional over $F$ algebraically closed, and let $ST = TS$, where the characteristic polynomial of $S$ has distinct roots.

(a) Show that every eigenvector of $S$ is an eigenvector for $T$.

(b) If $T$ is nilpotent, prove that $T = 0$.

Solution:

a) Since the characteristic polynomial of $S$ has distinct roots, all of the eigenspaces are 1-dimensional. Now suppose $Sv = \lambda v$, then $STv = TSv = \lambda (Tv)$, so $Tv$ is an eigenvalue of $S$ also with eigenvalue $\lambda$, so it is a multiple of $v$, say $Tv = \mu v$. So $v$ is an eigenvector of $T$.

b) Since $T$ has $n$ linearly independent eigenvectors by (a), we see $T$ is diagonalizable, hence its diagonalization must be the zero matrix (since that is the only nilpotent diagonal matrix). Hence $T$ is also the zero matrix.

2. (Jan 12.4): Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

(a) If $S, T$ are commuting linear operators on $V$, show that each eigenspace of $S$ is mapped onto itself by $T$.

(b) If $A_1, \cdots, A_k$ are operators which commute pairwise, show they have a common eigenvector in $V$.

(c) If $V$ has dimension $n$, show there exists a nested sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ where $\dim(V_j) = j$ and each $V_j$ is mapped onto itself by each of the operators $A_1, \cdots, A_k$.

Solution:

a) If $Sv = \lambda v$ then $S(Tv) = TSv = \lambda (Tv)$ so $Tv$ is also in the $\lambda$-eigenspace of $S$.

b) Induction on $k$: it is vacuously true for 1 operator. For the inductive step, let $\lambda$ be any eigenvalue of $A_k$ and let $W$ be the $\lambda$-eigenspace of $A_k$ (which is nonzero). By part (a), each of the operators $A_1, \cdots, A_{k-1}$ is a well-defined linear transformation on $W$, and they all commute with each other. So by the inductive hypothesis they have a common eigenvector $w$, which is also an eigenvector for $A_k$ by construction.

c) Inductive construction: Let $V_1 = \langle w \rangle$ where $w$ is the eigenvector from part (b). Now suppose we have constructed $V_{j-1}$ and consider the quotient space $V/V_{j-1}$. By hypothesis $A_1, \cdots, A_k$ are commuting linear operators on $V/V_{j-1}$ so by part (b) again, they have a common eigenvector $\bar{v} = v + V_{j-1}$. Then we can take $V_j = V_{j-1} \oplus \langle v \rangle$. It is then immediate that $A_i : V_j \to V_j$ and that $V_j$ is $j$-dimensional (since $\bar{v}$ is nonzero in $V/V_{j-1}$).
3. (Aug-05.4): Let $F$ be a field and $A, B$ nonsingular $3 \times 3$ matrices over $F$. Suppose $B^{-1}AB = 2A$.

(a) Find the characteristic of $F$.
(b) If $n$ is a positive or negative integer not divisible by 3, prove that $A^n$ has trace 0.
(c) Prove that the characteristic polynomial of $A$ is $X^3 - a$ for some $a \in F$.

**Solution:**

a) We have $\det(A) = \det(2A) = 8 \cdot \det(A)$ so since $A$ is nonsingular we see that $8 = 1$ in $F$, so the characteristic is 7.

b) We have $B^{-1}A^n B = 2^n A^n$ and trace is unaffected by conjugation, so $(2^n - 1) \cdot \text{tr}(A^n) = 0$. For $n \neq 0 \mod 3$, $2^n - 1 \neq 0 \mod 7$, so it is invertible in $F$; dividing by it gives $\text{tr}(A^n) = 0$.

c) By part (b), we see that $\text{tr}(A) = \text{tr}(A^2) = 0$. If $\alpha, \beta, \gamma$ are the eigenvalues of $A$, then $\text{tr}(A) = \alpha + \beta + \gamma$ and $\text{tr}(A^2) = \alpha^2 + \beta^2 + \gamma^2$, so we can write $\alpha \beta + \alpha \gamma + \beta \gamma = \frac{1}{2} \text{tr}(A^2) - \frac{1}{2} \text{tr}(A^2) = 0$. The characteristic polynomial is then $(x - \alpha)(x - \beta)(x - \gamma) = x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha \beta + \alpha \gamma + \beta \gamma)x - \alpha \beta \gamma = x^3 - \det(A)$.

alt) If $\alpha, \beta, \gamma$ are the eigenvalues of $A$, then since $2A$ is conjugate to $A$, $2\alpha, 2\beta, 2\gamma$ are also the eigenvalues of $A$, meaning that they are $\alpha, \beta, \gamma$, possibly permuted. Since $\alpha \neq 0$ we see $\alpha \neq 2\alpha$ (and the same for $\beta, \gamma$), so it is easy to see that the only possibility is that the permutation is a 3-cycle. Thus, up to swapping $\beta$ and $\gamma$, we have $\beta = 2\alpha, \gamma = 2\beta$, and $\alpha = 2\gamma$, meaning that the eigenvalues are $\alpha, 2\alpha, 4\alpha$, where $8\alpha = \alpha$. For (a), since $\alpha \neq 0$ we see the characteristic is 7. For (b), $\text{tr}(A^n) = \alpha^n(1^n + 2^n + 4^n)$, and $1^n + 2^n + 4^n$ is 0 mod 7 for any $n$ not divisible by 3. For (c), the characteristic polynomial is $(x - \alpha)(x - 2\alpha)(x - 4\alpha) = x^3 - (7\alpha)x^2 + (14\alpha)x - 8\alpha^3 = x^3 - 8\alpha^3$. (And the constant term is in $F$ since it is just $-1$ times the determinant of $A$.)

4. (Aug-06.4/85.4a): Let $S, T, M$ be $n \times n$ matrices over $\mathbb{C}$ with $SM = MT$.

(a) If $f(x)$ is the minimal polynomial of $T$, show $f(S)M = 0$.
(b) If $M \neq 0$, deduce that $S$ and $T$ have a common eigenvalue.
(c) Now let $S = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $T = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. Find a nonzero $M$ with $SM = MT$ and show that any such $M$ cannot be invertible.

**Solution:**

a) We have $S^n M = M T^n$, so $f(S)M = M f(T) = 0 \cdot 0 = 0$.

b) If $M \neq 0$ let $M v \in \ker f(S)$, so $\det(f(S)) = 0$. If we write $f(x) = \prod \lambda_i (x - \lambda_i)$ where the $\lambda_i$ are the eigenvalues of $T$, then $\det(f(S)) = \prod \lambda_i \det(S - \lambda_i I)$, hence $\det(S - \lambda_i I)$ is zero for some $\lambda_i$ — but this means $\lambda_i$ is an eigenvalue of both $S$ and $T$.

c) Routine computation shows we can take $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. $M$ cannot be invertible since if it were, $S$ and $T$ would be conjugate, but they’re not since their eigenvalues are clearly different.

**Remark** In fact, the converse of (b) is also true: If $S$ and $T$ have a common eigenvalue, then such a nonzero matrix $M$ does necessarily exist. To see this, observe that we can conjugate $S$ and $T$ independently (conjugate all three matrices, and then rescale $M$); so change variables to replace $S$ with its Jordan form and $T$ with the transpose of its Jordan form. We can then take $M$ to be the diagonal matrix with a 1 in the first entry and 0s elsewhere.
5. (Aug-06.4): Let $V$ be a nonzero finite dimensional vector space over $F$ and let $T : V \rightarrow V$ be a linear transformation. We say $T$ is regular if its characteristic polynomial and minimal polynomial are equal.

(a) If there exists a vector $v \in V$ such that $V$ is spanned by $v, T(v), T^2(v), \ldots$, prove that $T$ is regular.
(b) Assume that $T$ is regular and let $W$ be a subspace with $T(W) \subseteq W$. Show that $T_W$, the restriction of $T$ to $W$, and $T_{V/W}$, the induced action of $T$ on $V/W$, are both regular.

Solution:

a) Let $V$ be $n$-dimensional. We need to show that $T$ does not satisfy a nonzero polynomial of degree less than $n$, so suppose it did, say $f(x) = a_n x^n + \cdots + a_1 x + a_0$. Then in particular, $f(T)(v)$ is 0, and we can explicitly write $f(T)(v) = [T^n + a_n T^{n-1} + \cdots + a_1 T + a_0] v = a_n I^n + (a_n - a).$ If $v, Tv, \ldots$ are linearly-independent, as otherwise their span (which is the same as the span of $v, Tv, \ldots$ since any power above $T^{-1}$ is already dependent with the lower powers) would not be all of $V$. Therefore, $a_n = \cdots = a = 0$, and $f$ is the zero polynomial.

b) Choose a basis to make $T$ block-upper-triangular, say $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where $A$ corresponds to $T_W$ and $C$ corresponds to $T_{V/W}$. Then the characteristic polynomial $p_T(x)$ of $T$ is $\det(xI - T) = \begin{vmatrix} xI - A & -B \\ 0 & xI - C \end{vmatrix} = \det(xI - A) \cdot \det(xI - C)$, which is the product of the characteristic polynomials $p_W(x)$ of $T_W$ and $p_V(x)$ of $T_{V/W}$. So we have $m_W(x) : m_{V/W}(x) = p_W(x) \cdot p_{V/W}(x)$, $m_W(x) : p_W(x)$, and $m_{V/W}(x) : p_{V/W}(x)$, hence equality must hold and both $T_W$ and $T_{V/W}$ are regular.

6. (Jan-95.4) Let $A$ be an $n \times n$ matrix over an algebraically closed field $K$ and let $K[A]$ denote the $K$-linear span of $I, A, A^2, \ldots$. Show that $A$ is diagonalizable iff $K[A]$ contains no nonzero nilpotent element.

Solution: $A$ is diagonalizable iff the minimal polynomial $m(x)$ of $A$ has distinct roots. We see by definition of $m(x)$ that $K[A] \cong K[x]/m(x)$, and since $K$ is algebraically closed we can factor to get $m(x) = \prod (x - \lambda_i).$ We claim that any nilpotent element in $K[A]$ must be a multiple of $q(x) = \prod (x - \lambda_i)$; to see this merely observe that if $\prod (x - \beta)$ is nilpotent then the minimal polynomial must divide some power of it, hence each root of $m(x)$ divides it hence $q(x)$ divides it. Conversely, $q(x) = \prod (x - \lambda_i)$ is indeed nilpotent, and it will be zero in $K[x]/m(x)$ if and only if all eigenvalue multiplicities are equal to 1.

Note In fact $K$ does not even need to be algebraically closed as long as it is characteristic zero, for $q(x)$ above will actually have coefficients in $K$: if $f(x)$ is the characteristic polynomial of $A$, then the expression $q = f/\gcd(f, f')$ shows that $q$ is a quotient of polynomials with coefficients in $K$.

7. (Aug-03.4): Let $A$ be a real $n \times n$ matrix. We say $A$ is a “difference of two squares” if there exist real $n \times n$ matrices $B$ and $C$ for which $BC = CB = 0$ and $A = B^2 - C^2$.

(a) If $A$ is diagonal, show it is a difference of two squares.
(b) If $A$ is symmetric, show it is a difference of two squares.
(c) If $A$ is a difference of two squares with $B$ and $C$ as above, if $B$ has a nonzero real eigenvalue, prove that $A$ has a positive real eigenvalue.

Solution: Observe that we can conjugate each of $A, B, C$ by any invertible matrix $P$ and preserve the “difference of two squares” property. We will use this fact freely.

a) Recode the basis to put $A = \begin{pmatrix} D & 0 \\ 0 & -E \end{pmatrix}$ where $D$ and $E$ are diagonal matrices with nonnegative entries. Then we can take $B = \begin{pmatrix} D^{1/2} \\ 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 0 & E^{1/2} \end{pmatrix}$.

b) Real symmetric matrices are diagonalizable, so by the observation we can reduce to part (a) to see symmetric matrices are also a difference of two squares.

c) Let $\lambda \neq 0$ be the eigenvalue of $B$ with eigenvector $v$. Then $0v = CBv = \lambda(Cv)$, so $Cv = 0$. Then $Av = B^2v - C^2v = \lambda^2v$ so $A$ has an eigenvalue $\lambda^2 > 0$. 


8. (Aug-12.4): Let \( V \) be an \( n \)-dimensional \( K \)-vector space and \( T : V \to V \).

(a) Suppose there exists \( v \in V \) such that \( V \) is spanned by \( v, Tv, T^2v, \ldots \). Prove that the minimal polynomial of \( T \) equals the characteristic polynomial of \( T \).

(b) As a partial converse, suppose the characteristic polynomial of \( T \) has distinct roots in \( K \). Prove that there exists \( v \in V \) such that \( V \) is spanned by \( v, Tv, T^2v, \ldots \).

Solution:

a) If \( m(x) \) is the minimal polynomial of \( T \), then for any vector \( w \) we know that \( m(T)w = 0 \), so in particular \( m(T)v = 0 \). But since \( v, Tv, \ldots, T^{n-1}v \) are linearly independent, we see that the degree of \( m(x) \) must be \( \geq n \). But \( m(x) \) divides the characteristic polynomial, which has degree \( n \), so they must be equal since they are both monic.

b) Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( T \) with corresponding basis of eigenvectors \( v_1, \ldots, v_n \); by the given assumptions we know that the \( v_i \) are linearly independent. We claim that \( v = v_1 + \cdots + v_n \) has the desired property: to see this, suppose that \( p(T)v = 0 \): then we see that \( 0 = p(T)v = \sum p(\lambda_i)v_i \), so since the \( v_i \) are linearly independent we see that \( p(\lambda_i) = 0 \) for each \( \lambda_i \) - now since the eigenvalues of \( T \) are distinct, we see that \( p \) must be divisible by the characteristic polynomial of \( T \) hence have degree \( \geq n \). Thus, if \( p \) is any polynomial of degree \( < n \) we see that \( p(T)v \neq 0 \), so \( v, Tv, \ldots, T^{n-1}v \) are linearly independent, hence must span \( V \).

b-alt) Alternatively, since the characteristic polynomial of \( T \) has distinct roots in \( K \), this means \( T \) is diagonal with respect to an appropriate \( K \)-basis of \( V \): this follows by observing that the Jordan form of (any) matrix corresponding to \( T \) is diagonal, and then using the fact that if two matrices with \( K \)-coefficients are conjugate over \( \bar{K} \) then they are conjugate over \( K \) - this follows from properties of the rational canonical form. Then given such a diagonal matrix, it is easy to verify that \( v = [1, 1, 1, \ldots, 1] \) has the desired property.