1. (Jan-00.4): Let $A \in M_n(\mathbb{C})$ and assume that $A$ has rank 1.

(a) What are the possible Jordan canonical forms for $A$?
(b) For each of the forms in (a), find the characteristic and minimal polynomial of $A$.

2. (Jan-13.5): Let $W_n$ be the set of $n \times n$ complex matrices $C$ such that the equation $AB - BA = C$ has a solution in $n \times n$ matrices $A$ and $B$.

(a) Show that $W_n$ is closed under scalar multiplication and conjugation.
(b) Show that the identity matrix is not in $W_n$.
(c) Give a complete description of $W_2$.

3. (Jan-11.4): Let $V$ be a finite-dimensional $\mathbb{C}$-vector space and $T : V \to V$.

(a) Suppose $W$ is a subspace with $T(W) \subseteq W$. Show that the characteristic polynomial $f_S(x)$ of $S = T|_W$ divides the characteristic polynomial $f_T(x)$ of $T$ on $V$.
(b) Let $\lambda$ be a root of $f_T(x)$ of multiplicity $m$ and $V_\lambda = \{v \in V : T(v) = \lambda v\}$, Show that $1 \leq \dim_\mathbb{C} V_\lambda \leq m$.
(c) Find $(V, T, \lambda)$ such that $\lambda$ has multiplicity 5 as a root of $f_T(x)$ but $\dim_\mathbb{C} V_\lambda = 1$.

4. (Jan-14.2): Let $F$ be a field and $n$ a positive integer. Let $A \in M_{n \times n}(F)$ such that $A^n = 0$ but $A^{n-1} \neq 0$. Show that any $B \in M_{n \times n}(F)$ that commutes with $A$ is contained in the $F$-linear span of $I, A, A^2, \ldots, A^{n-1}$.

5. (Aug-82.7): Let $A \in M_n(\mathbb{C})$. Show that the following are equivalent:

(a) The ranks of $A$ and $A^2$ are equal.
(b) The multiplicity of 0 as a root of the minimal polynomial of $A$ is at most 1.
(c) There is an $n \times n$ matrix $X$ such that $AXA = A, XAX = X, AX = XA$.

6. (Aug-06.5): Let $F = \mathbb{F}_q$ and $M_2(F)$ be the ring of $2 \times 2$ matrices over $F$.

(a) If $A \in M_2(F)$ has equal eigenvalues in the algebraic closure of $F$, show that the eigenvalues of $A$ belong to $F$.
(b) Determine the number of nonzero nilpotent matrices in $M_2(F)$ as a function of $q$.

7. (Jan-10.4): Let $V$ be finite-dimensional over $F$ and $T : V \to V$, with characteristic polynomial $f(x) \in F[x]$.

(a) Show that $f(x)$ is irreducible in $F[x]$ iff there are no proper nonzero subspaces $W$ of $V$ with $T(W) \subseteq W$.
(b) If $f(x)$ is irreducible and $\text{char}(F) = 0$, show that $T$ is diagonalizable over $\bar{F}$.

8. (Jan-05.4): Let $F$ be an algebraically-closed field and $M_n(F)$ be the ring of $n \times n$ matrices over $F$. Describe those matrices $X \in M_n(F)$ such that all matrices that commute with $X$ are diagonalizable.