1. (Jan-97.4) Let $K$ be a field.
   (a) If $\text{char}(K) \neq 2$, show that $GL_n(K)$ has exactly $n$ conjugacy classes of elements of order 2.
   (b) If $\text{char}(K) = 2$, show that $GL_n(K)$ has exactly $\lceil n/2 \rceil$ conjugacy classes of elements of order 2.

2. (Aug-99.5) Let $f$ be the ideal generated by $\theta$. Find explicitly an element $g$.
   (b) If $\text{char}(R) = 2$, show that there is a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(M) = 0$.
   (c) Show that $Z$ is nilpotent, show that $\text{tr}(M)$ is an algebraic integer.

3. (Aug-94.5) Let $F$ be a field and $S = M_n(F)$.
   (a) If $s \in S$ is nilpotent, show that $\text{tr}(S) = 0$.
   (b) If $R$ is a ring (not necessarily commutative) and $\theta : R \rightarrow S$ is a surjective ring homomorphism, let $I$ be an ideal of $R$ such that every element of $I$ is the sum of nilpotent elements of $R$. Show that $\theta(I) = 0$.

4. (Aug-99.5) Let $F$ be a field, $f(x)$ and $g(y)$ be nonconstant polynomials in $R = F[x, y]$, and $I = (f(x), g(y))$, the ideal generated by $f$ and $g$.
   (a) Show that $I \neq R$.
   (b) If $f(x) = x - \alpha$ and $g(y) = y - \beta$ for $\alpha, \beta \in F$, show that $I$ is a maximal ideal.

5. (Jan-92.5) Let $\alpha_1, \ldots, \alpha_n$ be the roots of the polynomial $f(x) = 2x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$.
   (a) Show that $2\alpha_i$ is an algebraic integer for $1 \leq i \leq n$.
   (b) Show that $\mathbb{Z}[\alpha_1, \ldots, \alpha_n] \cap \mathbb{Q} \subseteq \mathbb{Z}[1/2]$.
   (c) If some $a_j$ with $0 \leq j \leq n - 1$ is odd, show that $1/2 \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n] \cap \mathbb{Q}$, and deduce that the latter intersection is $\mathbb{Z}[1/2]$. What happens if all $a_j$ are even?

6. (Jan-12.5) Let $K$ be a field where $-1$ is not a square, and let $G = GL_2(K)$.
   (a) If $g \in G$, show that $g$ has order 4 iff $\det(g) = 1$ and $\text{tr}(g) = 0$.
   (b) Find explicitly an element $g \in G$ of order 4.
   (c) Suppose there exist elements $a, b \in K$ with $a^2 + b^2 = -1$. Show that $G$ contains two elements $g, h$ of order 4 such that $gh$ also has order 4.

7. (Jan-96.5) Let $q$ be a prime power and $f(x) = \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1 \in \mathbb{F}_q[x]$.
   (a) If $f$ has a root in $\mathbb{F}_q$, show that $f$ splits completely over $\mathbb{F}_q$ and show that this happens precisely when $q \equiv 0, 1 \mod 5$.
   (b) If $f(x)$ has an irreducible monic factor $g(x)$ of degree 2, show that $g$ has constant term 1.
   (c) Factor $f(x)$ into quadratic factors when $q = 29$. 

2014 Algebra SEP ~ Grab Bag problems, by E. Dummit
8. (Jan-04.5) Let \( V \) be a finite-dimensional \( F \)-vector space and \( T : V \to V \). Assume that no nonzero proper subspace of \( V \) is mapped into itself by \( T \).

(a) If \( S \in F[T] \) is nonzero, show that \( \{ v \in V : Sv = 0 \} \) is the zero subspace.
(b) Prove that \( F[T] \) is a field.
(c) Show that \( |F[T] : F| = \dim_F V \).

9. (Jan-11.2) Let \( R \) be a commutative ring with 1, \( (a) = aR \), and \( P \) a prime ideal properly contained in \( (a) \).

(a) Show that \( P = aP \).
(b) If \( P \) is finitely generated, prove there exists \( b \in R \) with \((1 - ab)P = 0\).
(c) If \( R \) is a domain, conclude that either \( P = 0 \) or \( (a) = R \).

10. (Jan-07.5) Let \( A \) be an additive abelian group and \( B \) a subgroup. We say \( B \) is essential in \( A \) \( (B \text{ ess } A) \) if \( B \cap X \neq 0 \) for every nontrivial subgroup of \( A \).

(a) If \( B_1 \text{ ess } A_1 \) and \( B_2 \text{ ess } A_2 \) show that \( (B_1 \oplus B_2) \text{ ess } (A_1 \oplus A_2) \).
(b) If \( B \text{ ess } A \) and \( B \) has no nonzero elements of finite order, show \( A \) has no nonzero elements of finite order.
(c) If \( Q \text{ ess } A \) for some abelian group \( A \), show that \( A = Q \).

11. (Jan-08.4) Let \( V \) be a finite-dimensional vector space over \( F \) of characteristic \( p \), \( T : V \to V \), and \( W = \{ v \in V : Tv = v \} \). Further suppose \( T^p = I \) and \( \dim_F W = 1 \).

(a) Show that \( (T - I)^p = 0 \) and that \( \dim_F V \leq p \).
(b) If \( \dim_F V < p \) show that \( (T - I)^{p-1} = 0 \).
(c) If there exists \( v \in V \) with \( v + Tv + T^2v + \cdots + T^{p-1}v \neq 0 \), show \( \dim_F V = p \).

12. (Aug-11.2) Let \( R \) be a commutative ring with 1 and \( Q \) a primary ideal of \( R \). For any \( a \in R \setminus Q \), define the ideal \( I_a = \{ r \in R : ar \in Q \} \).

(a) Show that \( \rad(I_a) = \rad(Q) \).
(b) Show that \( I_a \) is a primary ideal of \( R \).
(c) If \( R \) is Noetherian, show that there exists an \( a \) such that that \( I_a \) is a prime ideal.

13. (Aug-07.2) Let \( R \) be a commutative integral domain that is integrally closed in its field of fractions \( F \).

(a) Suppose \( K \) is a field containing \( F \) and \( \alpha \in K \) is integral over \( R \). Show that the minimal monic polynomial of \( \alpha \) over \( F \) is in \( R[x] \).
(b) Let \( f(x) \in R[x] \) be monic. Show that \( f(x) \) is irreducible in \( R[x] \) iff it is irreducible in \( F[x] \).

14. (Jan-04.5) Let \( R \) be a ring with 1 and \( V = X \oplus Y \) for nonzero (right) \( R \)-modules \( X \) and \( Y \).

(a) Show that 0, \( X, Y, V \) are the only submodules of \( V \) iff \( X \) and \( Y \) are nonisomorphic simple \( R \)-modules.
(b) If \( X \) and \( Y \) are nonisomorphic simple \( R \)-modules, show that \( \End_R(V) \) is isomorphic to the direct sum of two division rings.