1. (Jan-06.2) Let $R$ be the subring of $\mathbb{Z}[x]$ consisting of all polynomials with zero $x$- and $x^2$-coefficients.

(a) Show that $\mathbb{Q}(x)$ is the field of fractions of $R$.
(b) Find the integral closure of $R$ in $\mathbb{Q}(x)$.
(c) Does there exist a polynomial $g(x) \in R$ such that $R$ is generated as a ring by 1 and $g(x)$?

2. (Aug-09.2/Jan-08.2a) Let $R \subseteq S$ be commutative rings with the same 1, and assume that every element of $S$ is integral over $R$.

(a) If $r \in R$ has an inverse in $S$, prove this inverse is in $R$.
(b) Suppose $R$ is a field and $s \in S$ is regular (i.e., if $sx = 0$ for some $x \in S$, then $x = 0$). Show that $s$ is invertible in $S$.
(c) If $P$ is a prime ideal of $S$, prove that $P$ is maximal in $S$ iff $R \cap P$ is maximal in $R$.

3. (Jan-01.3): Let $f(x) \in \mathbb{Z}[x]$ be monic and such that $f(\alpha) = f(2\alpha) = 0$ for some $\alpha \in \mathbb{C}$.

(a) Show that $f(0) \neq 1$.
(b) If $f$ is irreducible, prove $\alpha = 0$.

4. (Aug-12.5) Let $R$ be a not necessarily commutative ring with 1, such that $x^5 = x$ for every $x \in R$.

(a) Show that $J(R) = 0$.
(b) Now assume $R$ is right-Artinian. Prove that $R$ is a direct sum of division rings.
(c) Let $D$ be a division ring direct summand of $R$. If $F$ is any subfield of $D$, show that $F = \mathbb{F}_2$, $\mathbb{F}_3$, or $\mathbb{F}_5$.
(d) Deduce that $D$ above is isomorphic to $\mathbb{F}_2$, $\mathbb{F}_3$, or $\mathbb{F}_5$, and conclude that $R$ is commutative.

5. (Aug-04.2) Let $R$ be a ring with 1, $M$ be a finitely-generated (right) $R$-module, and $N \subset M$ a proper submodule of $M$.

(a) Prove that there exists a maximal submodule of $M$ containing $N$.
(b) Show that $N + MJ$ is a proper submodule of $M$, where $J = J(R)$ is the Jacobson radical of $R$.

6. (Aug-06.2) Let $R$ be a ring with 1 and $N$ a nil ideal of $R$ such that $R/N$ has no zero divisors.

(a) Show that the only idempotents of $R$ are 0 and 1.
(b) If $R/N$ is a division ring, show that every zero divisor in $R$ is nilpotent.
7. (Jan-14.1): Let $R$ be a commutative ring and $I$ an ideal of $R$.

(a) Show that the radical of $I$, $\text{rad}(I)$, is an ideal of $R$. (Recall that the radical is given by the set of all elements $x \in R$ such that there exists an integer $n$ such that $x^n \in I$.)

(b) Give an example of an ideal $I$ in $\mathbb{Q}[x,y]$ such that $I$ is non-principal but $\text{rad}(I)$ is principal.

(c) Suppose we try to define $\text{rad}(0)$ in $R = M_{2 \times 2}(\mathbb{R})$ to be the set of all elements $r \in R$ such that there exists an integer $n$ with $r^n = 0$. Show that this set $\text{rad}(0)$ is not an ideal of $R$.

8. (Aug-08.2) Let $S = \mathbb{Z} \oplus \mathbb{Z}$, and $R = \{(a, b) \in S : a \equiv b \text{ mod } 6\}$.

(a) Show that $R$ is a finitely-generated $\mathbb{Z}$-module and conclude that $R$ is a Noetherian ring.

(b) Prove that the ideal $P = \{(a, 0) \in R : a \equiv 0 \text{ mod } 6\}$ is prime in $R$.

(c) If $Q$ is a primary ideal of $R$ with $P = \text{rad}(Q)$, show that $Q = P$.

9. (Jan-12.2) Let $R$ be a commutative ring with $1$ and $Q$ be a primary ideal of $R$. Suppose that $Q = \bigcap X_i$ is a finite intersection of the ideals $X_i$.

(a) If each $X_i$ is prime, prove that $Q = X_j$ for some $j$. [Hint: Show that $Q$ is prime.]

(b) If $R$ is Noetherian and each $X_i$ is primary, and the radicals of the $X_i$ are distinct, prove again that $Q = X_j$ for some $j$.

10. (Aug-02.2) Let $R$ be a commutative ring with $1$ in which every proper ideal is primary.

(a) If $P$ is a prime ideal and $I$ is any ideal, show that either $I \subseteq P$ or $P = IP \subseteq I$.

(b) If $M$ is a maximal ideal of $R$, show that $M$ is the set of nonunits of $R$.

(c) Show that $J$ is prime in $R$ iff for all $r \in R$, $r^2 \in J$ implies $r \in J$. 