SOLUTIONS FOR JANUARY 2012

(1) (a) Let $P$ be a Sylow-11 subgroup of $G$. Then check that $|\text{Syl}_G(11)|$ can be either 1 or 56. If it is 56, then the normalizer of $P$ in $G$ has order $4312/56=77$. If it is 1, then the $P$ is normal in $G$, thus taking a subgroup $Q$ of order 7, $PQ$ has order 77.

(b) Let $S$ be the subgroup of order 77 (prove that it is actually isomorphic to $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}$), and let $Q$ be its Sylow-7 subgroup. Then let $T$ be a Sylow-7 subgroup of $G$ containing $Q$. Then $T$ is abelian, thus normalizing $Q$, so the normalizer of $Q$ contains a Sylow-7 and Sylow-11 subgroup of $G$, thus its index divides 8.

(c) Finally, if we have a subgroup of index dividing 8, and $G$ is simple, then $|G|$ divides 8!, which is not true in our case.

(2) (a) Assume all of $X_i$ are prime. Then by taking radicals, we obtain: $\sqrt{Q} = \bigcap_{i=1}^k \sqrt{X_i} = \bigcap_{i=1}^k X_i = Q$, thus $Q$ is prime (we are using that $Q$ is primary here). Then $\prod X_i \subseteq \bigcap X_i = Q$, thus there exists an $X_i \subseteq Q = \bigcap X_i$, which means that $Q = X_i$.

(b) Assume that $X_i$’s are primary and $R$ is noetherian. Also we can assume that $X_i$ are minimal decomposition of $Q$ (if it was not minimal, we can throw out some $X_i$’s). By the uniqueness of the decomposition, we actually obtain $k = 1$, and $Q = X_1$, which we wanted to prove.

(3) (a) The coefficients of the minimal polynomial are algebraic over $K$, since they are polynomials of the conjugates of $\alpha$. Thus they are in $A$, thus they are in $A \cap F = K$.

(b) By the primitive element theorem we have that $B = K[\beta]$. By a) the minimal polynomial of $\beta$ over $K$ and over $F$ are the same. Since the degree of the minimal polynomial is the same as the order of the field extension, we are done.

(c) Since $[A : K] = \sup\{[B : K]|K \subseteq B \subseteq A; [B : K] < \infty\}$, we are done.

(4) (a) Let $S_\lambda$ be the eigenspace of $S$ to eigenvalue $\lambda$. Then let $v \in S_\lambda$. Then $\lambda T v = T(\lambda v) = T(Sv) = S(Tv)$, thus $Tv \in S_\lambda$.

(b) If $k = 2$, then we saw previously that $T$ acted on $S_\lambda$. Thus $T$ has an eigenvector from $S_\lambda$, thus $T$ ans $S$ have a common eigenvector. If $k > 2$ we proceed by induction. So, let $V_e$ be the subspace of common eigenvectors of $A_1$, ..., $A_{k-1}$. Since $A_k$ acts on the eigenvectors, thus $A_k$ acts on $V_e$. Again, $A_k$ has an eigenvector from $V_e$, we are done.

(c) Let’s take a common eigenvector $v_1$, and let $V_1$ be its linear span. Now, $A_1$, ..., $A_k$ act on $V/V_1$, they commute, thus they have a common eigenvector. Let $v_2$ be any lift of this eigenvector to $V$. Now, $v_1$ and $v_2$ span $V_2$, we again take $V/V_2$, find eigenvector, lift it to $V$, ... It is easy to see that this sequence of $V_i$ satisfies the properties we want.
(5)  (a) Let’s pass to the algebraic closure of $K$. Then $G$ has a Jordan-form. Since $G$ is of order 4, one eigenvector should correspond to an eigenvalue $\lambda$ which is primitive forth root of unity. Since the determinant should be in $K$ and $K$ does not have a primitive forth root of unity, thus the other entry in the diagonal is also a primitive forth root of unity: $\lambda$ or $-\lambda$. Since the trace is also in $K$, and again $K$ does not have a primitive forth root of unity, thus the other entry in the diagonal equals $-\lambda$, so $\det G = 1$ and $\text{tr} G = 0$.

(This solution works also if $K$ is of characteristic 2)

(b) It is the operator: rotation by $90^\circ$. How does it look like as a matrix?

(c) Check that the following matrices are of order 4:

$$g = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, gh = \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}.$$