Algebra Qualifying Exam
January 1991

Do all 5 problems.

1. Let $G$ be a finite group having exactly $n$ Sylow $p$-subgroups for some prime $p$. Show that there exists a subgroup $H$ of the symmetric group $\text{Sym}_n$ such that $H$ also has exactly $n$ Sylow $p$-subgroups.

2. Let $R$ be a commutative integral domain and let $R[x_1, x_2, \ldots, x_n]$ be the polynomial ring over $R$ in the $n$ variables $x_1, x_2, \ldots, x_n$. If $a = (a_1, a_2, \ldots, a_n)$ is an $n$-tuple of elements of $R$, then there is an evaluation homomorphism $\varphi_a : R[x_1, x_2, \ldots, x_n] \to R$ given by $\varphi_a(f) = f(a_1, a_2, \ldots, a_n)$.
   a. (5 points) If $R$ is the field of complex numbers and if $I$ is a proper ideal of $R[x_1, x_2, \ldots, x_n]$, show that there exists an $n$-tuple $a = (a_1, a_2, \ldots, a_n)$ with $\varphi_a(I) \neq R$.
   b. (5 points) Now let $R$ be the ring of integers and let $I$ be the ideal of the polynomial ring $R[x]$ in one variable generated by 3 and $x^2 + 1$. Show that $I$ is a proper ideal of $R[x]$ but that $\varphi_a(I) = R$ for all $1$-tuples $a = (a)$ with $a \in R$.

3. Let $F \subseteq E$ be an extension of fields of characteristic $\neq 2$ and assume that the degree $|E : F| = 4$.
   a. (3 points) Show that $E = F[\alpha]$ for some $\alpha$.
   b. (2 points) If $E = F[\alpha]$, where $\alpha$ is a root of a polynomial of the form $x^4 + ax^2 + b \in F[x]$, prove that there exists an intermediate field properly between $E$ and $F$.
   c. (5 points) Now let $E = F[\beta]$ with no assumption on $\beta$ and let $L \supseteq E$ be a splitting field for the minimal polynomial of $\beta$ over $F$. If $\text{Gal}(L/F)$ is isomorphic to the symmetric group $\text{Sym}_4$, show that there is no intermediate field properly between $E$ and $F$.

4. Let $F$ be an algebraically closed field of prime characteristic $p$ and let $V$ be an $F$-vector space of dimension precisely $p$. Suppose $A$ and $B$ are linear operators on $V$ such that $AB - BA = B$. If $B$ is nonsingular, prove that $V$ has a basis $\{v_1, v_2, \ldots, v_p\}$ of eigenvectors of $A$ such that $Bv_i = v_{i+1}$ for $1 \leq i \leq p - 1$ and $Bv_p = \lambda v_1$ for some $0 \neq \lambda \in F$.

5. Let $G \neq (1)$ be a possibly infinite group whose subgroups are linearly ordered by inclusion. In other words, if $H$ and $K$ are subgroups of $G$, then either $H \subseteq K$ or $K \subseteq H$.
   a. (5 points) Prove that $G$ is an abelian group and that the orders of the elements of $G$ are all powers of the same prime $p$.
   b. (5 points) If $G_n = \{ g \in G \mid g^{p^n} = 1 \}$, prove that $|G_n| \leq p^n$. 
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Do all 5 problems.

1. Let $G$ be a finite group and fix a prime number $p$. Define the function $f$ on the set of subgroups $H \subseteq G$ by

$$f(H) = |\{ P \in \text{Syl}_p(G) \mid P \supseteq H \}|.$$ 

In other words, $f(H)$ is the number of Sylow $p$-subgroups of $G$ which contain $H$. Prove that if $f(H) > 0$, then $f(H) \equiv 1 \mod p$.

2. Let $F$ be a field and let $R$ be the ring of all $3 \times 3$ matrices over $F$ with $(3, 1)$ and $(3, 2)$ entry equal to 0. Thus,

$$R = \begin{pmatrix} F & F & F \\ F & F & F \\ 0 & 0 & F \end{pmatrix}.$$ 

Prove that the Jacobson radical of $R$ is a minimal left ideal, but not a minimal right ideal.

3. Let $F$ be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Suppose $E$ is a splitting field for $f(x)$ over $F$ and assume that there exists an element $\alpha \in E$ such that both $\alpha$ and $\alpha + 1$ are roots of $f(x)$.

i. Show that the characteristic of $F$ is not zero. (5 points)

ii. Prove that there exists a field $L$ between $F$ and $E$ such that the degree $|E : L|$ is equal to the characteristic of $F$. (5 points)

4. Let $V$ be a finite dimensional complex vector space and suppose $\langle \ , \ \rangle : V \times V \to \mathbb{C}$ is an inner product on $V$, that is, $\langle \ , \ \rangle$ is a positive definite Hermitian form on $V$.

i. Suppose $T : V \to V$ is a linear transformation such that $\langle Tv, v \rangle = 0$ for all $v \in V$. Prove that $V = 0$. (7 points)

ii. Does the result of part (i) hold if $V$ is assumed to be a real inner product space? Justify your answer. (3 points)

5. Let $\mathbb{Z}$ denote the ring of integers and let $\mathbb{Q}$ and $\mathbb{C}$ be the rational and complex fields, respectively. If $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$, then we let $\mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_n]$ denote the ring generated by these elements over $\mathbb{Z}$. In particular, note that $\mathbb{Z}[1/2]$ is the set of all rational numbers with denominator a power of 2. Now suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the integer polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbb{Z}[x]$ with $a_0 = 2$.

i. Prove that $2\alpha_i$ is an algebraic integer for all $i = 1, 2, \ldots, n$. (3 points)

ii. Show that $\mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_n] \cap \mathbb{Q} \subseteq \mathbb{Z}[1/2]$. (4 points)

iii. If some $a_j$ with $j \geq 1$ is odd, prove that $1/2 \in \mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_n] \cap \mathbb{Q}$ and deduce that the latter intersection is equal to $\mathbb{Z}[1/2]$. What happens if all $a_j$ are even? (3 points)
1. Let $G = A \times B$ be the internal direct product of finite subgroups $A$ and $B$. Suppose $H$ is a subgroup of $G$ with $A \cap H = 1$.
   i. Show that $B$ contains a subgroup isomorphic to $H$, but that $B$ need not contain $H$ in general. (5 points)
   ii. If $A$ and $B$ have relatively prime orders, prove that $H \subseteq B$. (5 points)

2. Let $K \subseteq E$ be an extension of fields and let $R$ be the subring of the polynomial ring $E[x]$ consisting of all polynomials with constant term in $K$. In other words, 
   
   $$R = K + Ex + Ex^2 + Ex^3 + \cdots$$

   Now let $I$ be a nonzero ideal of $R$ and let $m$ be the minimal degree of the nonzero elements of $I$. Define 
   
   $$I_m = \{ f(x) \in I \mid \deg f(x) = m \} \cup \{ 0 \}$$

   so that $I_m$ is clearly a nonzero $K$-subspace of $R$.
   i. If $\dim_K I_m = 1$, prove that $I$ is a principal ideal. (4 points)
   ii. If $\dim_K I_m > 1$, show that $I_m$ contains a nonzero polynomial with constant term equal to 0. (2 points)
   iii. If $I$ is a prime ideal which is not principal, prove that $m = 1$. (4 points)

3. Let $\mathbb{Q}$ be the field of rational numbers and let $K = \mathbb{Q}[\sqrt{2}]$. Suppose $f(x) \in \mathbb{Q}[x]$ is a monic irreducible polynomial of odd degree $n \geq 1$ and notice that $f(x + \sqrt{2})$ is a monic polynomial of degree $n$ in $K[x]$.
   i. Show that the coefficient of $x^{n-1}$ in $f(x + \sqrt{2})$ is not rational. (2 points)
   ii. Show that the polynomial $f(x + \sqrt{2})$ is irreducible in $K[x]$. (4 points)
   iii. Prove that the polynomial $g(x) = f(x + \sqrt{2})f(x - \sqrt{2})$ is irreducible in the ring $\mathbb{Q}[x]$. (4 points)

4. Let $V \neq 0$ be a finite-dimensional vector space over the complex numbers $\mathbb{C}$ and let $X, Y, Z$ be linear operators on $V$ which satisfy

   $$XY - YX = Z \quad XZ = ZX \quad YZ = ZY.$$ 

   If $V$ has no proper subspace invariant under all three operators, prove that $\dim \mathbb{C} V = 1$.

5. Let $V$ be a vector space over the rational numbers $\mathbb{Q}$ and let $v_1, v_2, \ldots, v_n \in V$. Show that there exist elements $w_1, w_2, \ldots, w_m \in V$ which are linearly independent over $\mathbb{Q}$ and which satisfy

   $$\sum_{i=1}^{n} v_i Z = \sum_{j=1}^{m} w_j Z$$

   where $Z$ is the ring of integers.
1. A finite group is said to be **perfect** if it has no nontrivial abelian homomorphic image.
   i. Show that a perfect group has no nontrivial solvable homomorphic image. (3 points)
   ii. Let $H \triangleleft G$ with $G/H$ perfect. If $\theta: G \to S$ is a homomorphism from $G$ to a solvable group $S$ and if $N = \ker \theta$, prove that $G = NH$ and deduce that $\theta(H) = \theta(G)$. (7 points)

2. Let $R$ be a ring and let $V$ be a right $R$-module. Assume that every simple submodule of $V$ is a direct summand of $V$.
   i. If $W$ is any submodule of $V$, show that any simple submodule of $W$ is a direct summand of $W$. (5 points)
   ii. If $V$ is an Artinian module, that is if its submodules satisfy the minimal condition, prove that $V$ is a direct sum of finitely many simple submodules. (5 points)

3. Let $\alpha$ be the real positive 16th root of 3 and consider the field $F = Q[\alpha]$ generated by $\alpha$ over the rationals $Q$. Notice that we have the chain of intermediate fields

   $$Q \subseteq Q[\alpha^8] \subseteq Q[\alpha^4] \subseteq Q[\alpha^2] \subseteq Q[\alpha] = F.$$ 

   i. Compute the degrees of these five intermediate fields over $Q$ and conclude that these fields are all distinct. (4 points)
   ii. Show that every intermediate field between $Q$ and $F$ is one of the above. Hint. If $Q \subseteq K \subseteq F$, consider the constant term of the minimal polynomial of $\alpha$ over $K$. (6 points)

4. Let $X$ be a subspace of $M_n(C)$, the $C$-vector space of all $n \times n$ complex matrices. Assume that every nonzero matrix in $X$ is invertible. Prove that $\dim_C X \leq 1$.

5. Let $E$ be an algebraic extension of the rational numbers $Q$ and let $\alpha \in E$.
   i. Prove that there exists a nonzero integer $n \in Z$ such that $n\alpha$ is an algebraic integer. (4 points)
   ii. Show that $Z[\alpha]$ does not contain $Q$ and hence conclude that $Z[\alpha]$ is not a field. (6 points)
1. Let $P$ be a Sylow $p$-subgroup of a finite group $G$ and let $N = N_G(P)$ be its normalizer.
   i. Show that $N$ is not contained in any proper normal subgroup of $G$. (5 points)
   ii. If the commutator subgroup $G'$ is abelian, prove that $G' \cap N$ is normal in $G$. (5 points)

2. Let $R$ be a commutative Noetherian domain and suppose that $P$ is the unique nonzero prime ideal of $R$.
   i. Show that every element of $R$ not in $P$ is a unit of $R$. (3 points)
   ii. If $Q$ is a nonzero ideal of $R$, prove that $Q$ is primary and that $Q \supseteq P^n$ for some integer $n \geq 1$. (3 points)
   iii. If $P = (\pi)$ is principal, prove that every nonzero element of $R$ is a product of a unit and a power of $\pi$. (4 points)

3. Let $\mathbb{Q}$ be the field of rational numbers and let $f(x) = x^8 + x^4 + 1$ be a polynomial in $\mathbb{Q}[x]$. Suppose $E$ is a splitting field for $f(x)$ over $\mathbb{Q}$ and set $G = \text{Gal}(E/\mathbb{Q})$.
   i. Find $|E:\mathbb{Q}|$ and determine the Galois group $G$ up to isomorphism. (5 points)
   ii. If $\Omega \subset E$ is the set of roots of $f(x)$, find the number of orbits for the action of $G$ on $\Omega$. (5 points)

4. Let $A$ be an $n \times n$ matrix over an algebraically closed field $K$ and let $K[A]$ denote the $K$-linear span of the matrices $I = A^0, A, A^2, A^3, \ldots$. Show that $A$ is diagonalizable if and only if $K[A]$ contains no nonzero nilpotent element.

5. Let $G$ be a (not necessarily finite) group and denote the operation in $G$ by multiplication. Let $\mathbb{Z}[G]$ denote the group ring of $G$ over the integers $\mathbb{Z}$. Thus, every element of $\mathbb{Z}[G]$ is a finite $\mathbb{Z}$-linear combination of elements of $G$, and the multiplication in $\mathbb{Z}[G]$ is built naturally from the multiplication in $G$. Let $I$ be a right ideal of $\mathbb{Z}[G]$ and define

$$\text{gp}(I) = \{ g \in G \mid 1 - g \in I \}.$$ 

   i. Prove that $\text{gp}(I)$ is a subgroup of $G$. (4 points)
   ii. If $I$ is a 2-sided ideal of $\mathbb{Z}[G]$, show that $\text{gp}(I)$ is normal in $G$. (3 points)
   iii. If $\text{gp}(I) = G$, prove that $I$ is a 2-sided ideal of $\mathbb{Z}[G]$. (3 points)
1. A finite group $G$ is said to have property $\ast$ if there exists a conjugacy class $K$ of $G$ such $G$ is generated by the elements of $K$.
   i. If $G$ has property $\ast$, prove that $G/G'$ is cyclic, where $G'$ is the commutator subgroup of $G$. (3 points)
   ii. Show that $G$ has property $\ast$ if and only if $G$ is not the set-theoretic union of its proper normal subgroups. (3 points)
   iii. Suppose $G = N \times M$ where $N$ is nonabelian simple and $M$ has property $\ast$. Prove that $G$ has property $\ast$. (Hint. First show that every normal subgroup of $G$ that does not contain $N$ must be contained in $M$.) (4 points)

2. Let $R$ be a commutative ring with 1 and suppose that $M$ is an ideal of $R$.
   i. If $M$ is both maximal and principal, show that there is no ideal $I$ of $R$ satisfying $M > I > M^2$, where $>$ denotes strict inclusion. (6 points)
   ii. Give examples to show that neither of the two conditions on $M$ in part (i) can be removed. (4 points)

3. Let $Q \subseteq L \subseteq E$ be fields with $Q$ the rational numbers and with $|E : L| < \infty$. Let $K$ be the subfield of $E$ consisting of all those elements of $E$ which are algebraic over $Q$, and assume that $K \cap L = Q$.
   i. If $\alpha \in K$, show that its minimal polynomial over $Q$ is irreducible over $L$ and deduce that $|Q[\alpha] : Q| \leq |E : L|$. (5 points)
   ii. Show that $|K : Q| \leq |E : L|$. (Hint. Start with a subfield $M$ of $K$ maximal with the property that $|M : Q| \leq |E : L|$.) (5 points)

4. Let $V$ be a vector space over a field $K$ and let $S$ and $T$ be $K$-linear operators on $V$. Suppose that $S$ is one-to-one, that $T(v) = 0$ for some $0 \neq v \in V$, and that $TS - ST = S$.
   i. For every integer $n \geq 0$, prove that $S^n(v)$ is an eigenvector for $T$ and determine the corresponding eigenvalue. (4 points)
   ii. If $K$ has characteristic 0, prove that $\dim_K V = \infty$. (4 points)
   iii. If $K$ has characteristic $p > 0$, then $\dim_K V$ can be finite. Give a concrete example of such a finite-dimensional situation when $p = 3$. (2 points)

5. Let $q$ be a prime power and let $F$ be the finite field of size $q$. Let
   $$f(x) = \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1 \in F[x].$$
   i. If $f(x)$ has a root in $F$, show that $f(x)$ splits completely over $F$, and prove that this happens precisely when $q \equiv 0$ or 1 mod 5. (6 points)
   ii. If $f(x)$ has an irreducible monic factor $g(x)$ of degree 2, show that $g(x)$ has constant term equal to 1. (2 points)
   iii. Factor $f(x)$ explicitly into irreducible quadratic factors when $q = 29$. (2 points)
1. Let $G$ be a finite group having the property that for every choice of two subgroups $X \subseteq G$ and $Y \subseteq G$, either $X \cap Y = 1$ or $X \subseteq Y$ or $Y \subseteq X$.
   i. If $H \subseteq G$, show that either $|H|$ is a prime power or else that $|H|$ and $|G : H|$ are relatively prime. (4 points)
   ii. If $1 < N \triangleleft G$, prove that $G/N$ is nilpotent. (2 points)
   iii. If $N \triangleleft G$ and $N \neq G$, show that $N$ is nilpotent. (4 points)

2. Let $R$ be a ring, let $V$ be a right $R$-module, and suppose that $V = V_1 \oplus V_2 \oplus V_3 \oplus \cdots$ is the (internal) direct sum of its submodules $V_1, V_2, V_3, \ldots$. Show that $V$ is an Artinian module if and only if each $V_i$ is Artinian and only finitely many of the $V_i$'s are nonzero.

3. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 5 over the rational numbers $\mathbb{Q}$ that is not solvable by radicals, and let $S$ be the splitting field of $f(x)$ over $\mathbb{Q}$ which is contained in the complex numbers.
   i. Show that there exists at most one subfield $E$ of $S$ such that $|E : \mathbb{Q}| = 2$. (7 points)
   ii. If $\alpha, \beta \in S$ are irrational elements which satisfy $\alpha^2 \in \mathbb{Q}$ and $\beta^2 \in \mathbb{Q}$, prove that $\alpha\beta \in \mathbb{Q}$. (3 points)

4. If $K$ is a field, then the general linear group $G = \text{GL}_n(K)$ is the multiplicative group of $n \times n$ invertible matrices over $K$.
   i. If the characteristic of $K$ is not equal to 2, show that $G$ has precisely $n$ conjugacy classes of elements of order 2. (5 points)
   ii. If $\text{char } K = 2$, show that $G$ has precisely $\lfloor n/2 \rfloor$ (the greatest integer in $n/2$) conjugacy classes of elements of order 2. (5 points)

5. Let $S$ be a commutative integral domain and let $R$ be a subring of $S$ with the same identity 1. Suppose that there exist finitely many elements $s_1, s_2, \ldots, s_n \in S$ such that $S = s_1 R + s_2 R + \cdots + s_n R$. Show that $R$ is a field if and only if $S$ is a field.
1. Fix a prime $p$ and let $G$ be a finite group with the property that every nonidentity $p$-subgroup of $G$ is contained in a unique Sylow $p$-subgroup of $G$. Suppose $N \triangleleft G$ and $|N|$ is divisible by $p$.
   
i. If $P$ and $Q$ are Sylow $p$-subgroups of $G$, show that $Q = P^n$ for some element $n \in N$. (6 points)
   
   ii. Prove that $G/N$ has a unique Sylow $p$-subgroup. (4 points)

2. Let $R$ be a commutative domain and write $(a)$ for the principal ideal generated by $a \in R$. Recall that an element of $R$ is said to be irreducible if it is nonzero, not a unit, and has no proper factorization.
   
i. Show that $(a) \subseteq (b)$ if and only if $b \mid a$, and that $(a) = (b)$ if and only if $b = au$ for some unit $u \in R$. (2 points)
   
   ii. If $R$ is a UFD (unique factorization domain), prove that the set of principal ideals of $R$ satisfies the maximal condition. (4 points)
   
   iii. If the set of principal ideals of $R$ satisfies the maximal condition, show that every nonzero, nonunit element of $R$ can be written as a finite product of irreducible elements. (4 points)

3. Let $p$ be a prime, let $F \subseteq K$ be fields of characteristic 0, and assume that $F$ contains a primitive $p$th root of unity. Fix $a \in K$.
   
i. Prove that there exists a field $E \supseteq K$ such that $E$ contains a $p$th root of $a$ and $|E : K| = 1$ or $p$. (4 points)
   
   ii. Now assume that $K$ is a finite degree Galois extension of $F$. Show that there exists a field $E \supseteq K$ such that $E$ contains a $p$th root of $a$, $E$ is Galois over $F$, and $|E : K|$ is a power of $p$. (6 points)

4. Let $V$ be a finite dimensional vector space over a field of characteristic 0. Suppose $T : V \to V$ is a linear operator such that the trace $\text{tr} T^k = 0$ for all integers $k \geq 1$.
   
i. Show that the constant term of the characteristic polynomial of $T$ is zero, and deduce that $T(V) \neq V$. (5 points)
   
   ii. Let $S$ denote the restriction of $T$ to the subspace $T(V)$, so that $S$ is a linear operator on $T(V)$. Prove that $\text{tr} S^k = 0$ for all integers $k \geq 1$. (4 points)
   
   iii. Show that $T$ is nilpotent. (1 point)

5. Let $G$ be a (not necessarily finite) group and let $\theta : G \to G$ be a homomorphism such that $\theta^n(G) = \{1\}$ for some integer $n \geq 1$.
   
i. If the kernel of $\theta$ is finite, prove that the kernel of $\theta^2$ is finite, and deduce that $G$ is finite. (5 points)
   
   ii. If $\theta(G)$ has finite index in $G$, prove that $\theta^2(G)$ has finite index in $G$, and deduce that $G$ is finite. (5 points)
1. Let $G$ be a finite group and let $A \subseteq G$ be a maximal (proper) subgroup. Assume that $A$ is abelian, that $|G : A| = p^n$ for some prime $p$, and that $A$ contains no nonidentity normal subgroup of $G$.
   i. Show that $p$ does not divide $|A|$. (4 points)
   ii. Prove that the set $S$ of elements of $G$ not conjugate to any nonidentity element of $A$ has cardinality precisely $p^n$. (5 points)
   iii. Show that $G$ is not simple. (1 point)

2. Let $R$ be a ring with 1, and recall that $R$ is naturally a right $R$-module with respect to right multiplication. We denote this right regular $R$-module by $R_R$.
   i. Prove that $R$ is a division ring if and only if $R_R$ is a simple $R$-module. (5 points)
   ii. Prove that $R$ is a division ring if and only if every nonzero right $R$-module contains a submodule isomorphic to $R_R$. (5 points)

3. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial over the rational numbers $\mathbb{Q}$, and let $\alpha$ and $\beta$ be roots of $f(x)$ in the complex numbers $\mathbb{C}$. Suppose $\mathbb{Q} \subseteq E \subseteq \mathbb{C}$ where $E$ is a (finite) Galois extension of $\mathbb{Q}$.
   i. Show that $\mathbb{Q}[\alpha] \cap E$ is isomorphic to $\mathbb{Q}[\beta] \cap E$. (5 points)
   ii. Now assume that $E = \mathbb{Q}[\varepsilon]$ where $\varepsilon$ is a root of unity. Prove that $\mathbb{Q}[\alpha] \cap E = \mathbb{Q}[\beta] \cap E$. (5 points)

4. Let $V$ be a finite-dimensional vector space over an algebraically closed field $F$, and let $S$ and $T$ be commuting linear operators on $V$. Assume that the characteristic polynomial of $S$ has distinct roots.
   i. Show that every eigenvector of $S$ is an eigenvector for $T$. (5 points)
   ii. If $T$ is nilpotent, prove that $T = 0$. (5 points)

5. Let $R$ be a ring with 1 and let $V$ be a right $R$-module. Assume that $M_1, M_2, \ldots, M_n$ are finitely many $R$-submodules of $V$ with $M_1 \cap M_2 \cap \cdots \cap M_n = 0$, and let $W$ be the (external) direct sum $W = V/M_1 \oplus V/M_2 \oplus \cdots \oplus V/M_n$.
   i. Show that $V$ is isomorphic to an $R$-submodule of $W$. (4 points)
   ii. Now suppose in addition that the modules $V/M_i$ are simple and pairwise nonisomorphic. Prove that $V$ is isomorphic to $W$. (Hint. First observe that $W$ has a composition series of length $n$.) (6 points)
Algebra Qualifying Exam
January 2000

Do all 5 problems.

1. Let $G$ be a group of order $2^4 \cdot 5^3 \cdot 11$ and let $H$ be a group of order $5^3 \cdot 11$.
   a. Show that $H$ has a normal Sylow 11-subgroup. (2 points)
   b. If the number of Sylow 5-subgroups of $G$ is (strictly) less than 16, prove that $G$ has a proper normal subgroup of order divisible by 5. (4 points)
   c. If $G$ has exactly sixteen Sylow 5-subgroups, show that $G$ has a normal Sylow 11-subgroup. (4 points)

2. Let $R$ be a (not necessarily commutative) ring with 1 and suppose that $R$ can be written as the sum $R = \sum_{i=1}^{m} I_i$, where the $I_i$ are finitely many (two-sided) ideals of $R$ satisfying $I_i \cap I_j = 0$ whenever $i \neq j$.
   a. Prove that, for every simple right $R$-module $M$, there exists a unique subscript $k$ such that $MI_k \neq 0$. (5 points)
   b. Show that if $i \neq j$, then every right $R$-module homomorphism $\theta: I_i \to I_j$ is the zero map. (5 points)

3. Let $L/K$ be a finite degree Galois extension of fields with Galois group given by $\text{Gal}(L/K) = G$, and let $E$ be an intermediate field. Then $E$ is said to be a 2-tower over $K$ if there exists a chain of fields $K = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$ such that $|E_i : E_{i-1}| = 2$ for all $i = 1, 2, \ldots, n$.
   a. If $G$ is abelian, prove that $E$ is a 2-tower over $K$ if and only if the degree $|E : K|$ is a power of 2. (7 points)
   b. Show by example that the characterization of 2-towers given in part (a) is false if $G$ is allowed to be a nonabelian group. (3 points)

4. Let $A$ be an $n \times n$ matrix over the complex numbers and assume that the rank of $A$ is equal to 1.
   a. What are the possible Jordan canonical forms for $A$? Justify your answer. (5 points)
   b. For each of the forms obtained in part (a), compute the characteristic polynomial of $A$ and the minimal polynomial of $A$. (5 points)

5. Let $R = F[x, y]$ be the polynomial ring over the field $F$ in the two indeterminates $x$ and $y$, and let $I = xR$ be the principal ideal of $R$ generated by $x$. Define $S = F + I$, so that $S$ is a subring of $R$, and observe that $I$ is an ideal of $S$.
   a. Show that $I$ is not finitely generated as an ideal of $S$. (5 points)
   b. Prove that there are infinitely many ideals of $S$ that are not ideals of $R$. (5 points)
Algebra Qualifying Exam
January 2001

Do all 5 problems.

1. Let $X$ and $Y$ be distinct subgroups of a finite group $G$. We say that $X$ and $Y$ are a weird pair if $|X| = |Y|$ and if no subgroup of $G$ other than $X$ and $Y$ has this same order.
   a. If $G$ is a group having a weird pair of subgroups, show that some subgroup of $G$ has a weird pair of normal subgroups. (3 points)
   b. If $G = A \times B$ is a direct product of solvable groups, show that the subgroups $A \times 1$ and $1 \times B$ cannot be a weird pair. (4 points)
   c. Show that a solvable group cannot contain a weird pair of subgroups. (3 points)

2. Let $R$ be a ring with 1. Recall that an ideal $P$ of $R$ is said to be (right) primitive if there exists a simple right $R$-module $W$ with $P = \{ r \in R \mid Wr = 0 \}$. Furthermore, we recall that the Jacobson radical, $\text{Jrad}(R)$, of $R$ is defined to be the intersection of all primitive ideals of $R$.
   a. Let $V$ be a right $R$-module having a composition series of length $n$ and suppose that $R$ acts faithfully on $V$. Show that $J = \text{Jrad}(R)$ is an intersection of $n$ primitive ideals of $R$ and that $J^n = 0$. (7 points)
   b. Give an example of the situation in part (a) with $n = 2$ and with $\text{Jrad}(R) \neq 0$. Justify your answer. (3 points)

3. Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial with integer coefficients and suppose that $f(\alpha) = 0 = f(2\alpha)$ for some complex number $\alpha$.
   a. Show that $f(0)$, the constant term of $f$, is not equal to 1. (5 points)
   b. If $f$ is irreducible, prove that $\alpha = 0$. (5 points)

4. Let $A$ be an $n \times n$ matrix over the complex numbers and let $A^*$ denote the conjugate transpose of $A$.
   a. Prove that all eigenvalues of the product matrix $A^*A$ are real and nonnegative. (6 points)
   b. If $I$ is the $n \times n$ complex identity matrix, show that $\det(I + A^*A)$ is real and positive. (4 points)

5. Let $V$ be a finite-dimensional vector space over some field $F$ and let $T: V \to V$ be a linear operator. Write $F[T]$ to denote the ring of all linear operators on $V$ that can be expressed as polynomials in $T$. Assume that no nonzero proper subspace of $V$ is mapped into itself by $T$.
   a. If $0 \neq S \in F[T]$, show that $\{ v \in V \mid vS = 0 \}$ is the zero subspace. (3 points)
   b. Prove that $F[T]$ is a field. (4 points)
   c. Show that $[F[T] : F] = \dim_F V$. (3 points)
Algebra Qualifying Exam
January 2002

Do all 5 problems.

1. Let $N$ be a normal subgroup of the finite group $G$. A subgroup $H$ of $G$ is said to be a complement for $N$ in $G$ if $NH = G$ and $N \cap H = 1$.
   a. Show that all complements for $N$ in $G$ are isomorphic. (2 points)
   b. If $N$ has a complement in $G$ that is a $p$-group for some prime $p$, prove that every Sylow $p$-subgroup of $G$ contains a complement for $N$. (3 points)
   c. Assume that the center of $N$ is trivial, that is equal to the identity subgroup, and that every automorphism of $N$ is inner. Prove that $N$ has a unique complement $H$ that is normal in $G$. (5 points)

2. Let $R$ be a commutative integral domain with 1, and assume that $R$ is integrally closed in its field of fractions. Let $R[x]$ denote the ring of polynomials over $R$ in the variable $x$.
   a. Let $S \supseteq R$, where $S$ is a commutative integral domain with the same 1, and let $s \in S$ be an element integral over $R$. If $I$ is the ideal of $R[x]$ consisting of all polynomials $f(x)$ with $f(s) = 0$, prove that $I$ is principal. (5 points)
   b. Let $I$ be a prime ideal of $R[x]$ containing a monic polynomial. If $I \cap R = 0$, prove that $I$ is principal. (5 points)

3. Let $K$ be a field of prime characteristic $p$ and let $F = K(t)$ be the rational function field over $K$ in the indeterminate $t$. Write $f(x) = x^{2p} - tx^p + t \in F[x]$.
   a. Show that $f(x)$ is an irreducible polynomial in $F[x]$. (3 points)
   b. Let $E = F[s]$, where $s$ is a root of the polynomial $x^p - t \in F[x]$. If $L$ is the splitting field of $f(x)$ over $E$, prove that $|L : E| \leq 2$. (4 points)
   c. Show that $L = F[\alpha]$, where $\alpha$ is a root of $f(x)$. (3 points)

4. Let $V$ be an $n$-dimensional vector space over the field $K$ and let $T : V \to V$ be a linear operator. Write $K[T]$ to denote the ring of all linear operators on $V$ that can be expressed as polynomials in $T$, and let $C$ denote the $K$-vector space of all linear operators on $V$ that commute with $T$. Assume that there exists a vector $v_0 \in V$ that is contained in no proper $T$-invariant subspace of $V$.
   a. Show that $v_0 K[T] = V$ and deduce that $\dim_K K[T] \geq n$. (3 points)
   b. If $S \in C$ with $v_0 S = 0$, show that $S = 0$. Deduce that $\dim_K C \leq n$. (3 points)
   c. Show that $K[T] = C$, and deduce that the minimal polynomial of $T$ has degree equal to $n$. (4 points)
5. Let $R$ be a ring with 1, let $V$ be a right $R$-module with a composition series, and let $E = \text{End}_R(V)$ be the ring of $R$-endomorphisms of $V$.

a. If $\theta: V \to V$ is an element of $E$, prove that $\theta$ is one-to-one if and only if it is onto and hence if and only if it is invertible in $E$. (4 points)

b. If $V$ has a unique minimal $R$-submodule $U$, prove that $E$ has a unique maximal ideal $I$ and that every element of $E \setminus I$ is invertible. Be sure to verify that $I$ is indeed an ideal. (3 points)

c. Again, let $V$ have unique minimal submodule $U$, and suppose in addition that $U$ has multiplicity 1 as an $R$-composition factor of $V$. Prove that $E$ is a division ring. (3 points)
Algebra Qualifying Exam
January 2003

Do all 5 problems. In the following, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{C}$ is the field of complex numbers.

1. Let $N$ be a normal subgroup of the finite group $G$ and suppose that $G/N$ is a $p$-group for some prime $p$.
   a. If $N \subseteq Z(G)$, the center of $G$, show that the commutator subgroup $G'$ of $G$ is a $p$-group. (5 points)
   b. Now assume that $N$ is cyclic (but not necessarily central in $G$). Prove that $N \cap G' \subseteq Z(G')$ and deduce that $G''$ is a $p$-group. (5 points)

2. Let $R$ be a commutative integral domain with 1. A nonzero, nonunit element $s \in R$ is said to be “special” if, for every element $a \in R$, there exist $q, r \in R$ with $a = qs + r$ and such that $r$ is either 0 or a unit of $R$.
   a. If $s \in R$ is special, prove that the principal ideal $(s)$ generated by $s$ is maximal in $R$. (3 points)
   b. Show that every polynomial in $\mathbb{Q}[X]$ of degree 1 is special in $\mathbb{Q}[X]$. (2 points)
   c. Prove that there are no special elements in the polynomial ring $\mathbb{Z}[X]$. (Hint. Apply the definition of special with $a = 2$ and with $a = X$.) (5 points)

3. Let $F$ be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$, where $F/\mathbb{Q}$ is a finite Galois extension. Let $\alpha \in F$ and let $f(X) \in \mathbb{Q}[X]$ be its minimal monic polynomial. Assume that $1 = |\alpha|$, the absolute value of $\alpha$, and that $\text{Gal}(F/\mathbb{Q})$ is abelian.
   a. Show that $F$ is closed under complex conjugation. (2 points)
   b. Prove that $|\beta| = 1$ for every complex root $\beta$ of $f(X)$. (3 points)
   c. Writing $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$, show that $|a_i| \leq 2^n$ for all $i$ with $0 \leq i < n$. (2 points)
   d. Prove that $F$ contains only finitely many algebraic integers having absolute value 1 and deduce that each of these is a root of unity. (3 points)

(over)
4. Let $V$ be vector space over the field $K$ and let $(\ ,\ ): V \times V \to K$ be a bilinear form on $V$.
   a. If $V$ is finite dimensional and if $W$ is a proper subspace of $V$, show that there exists a nonzero vector $v \in V$ with $(w,v) = 0$ for all $w \in W$. (5 points)
   b. Now let $V$ have an infinite basis $\mathcal{B}$ and let $(\ ,\ )$ be the unique bilinear form such that, for all $a, b \in \mathcal{B}$, we have $(a,b) = 0$ if $a \neq b$ and $(a,b) = 1$ if $a = b$. If $W$ is the subspace of $V$ spanned by all vectors of the form $a - b$ with $a, b \in \mathcal{B}$, show that $W$ is a proper subspace of $V$ and that there is no nonzero vector $v \in V$ with $(w,v) = 0$ for all $w \in W$. (5 points)

5. Let $R$ be a ring with 1. We say that a right $R$-module $W$ is “infinitely generated” if it is not finitely generated as an $R$-module.
   a. Let $V$ be a right $R$-module and let $W$ be a submodule of $V$. If $W$ is infinitely generated, prove that there exists a submodule $M$ with $W \subseteq M \subseteq V$ such that $M$ is infinitely generated, but such that all submodules of $V$ properly containing $M$ are finitely generated. (5 points)
   b. If $R$ is right Noetherian, show that $M = V$ in the above situation. (2 points)
   c. If $R$ is not right Noetherian, show that it is possible to choose $V$ and $W$ as in part (a) so that $M \neq V$. (3 points)
Algebra Qualifying Exam
January 2004

Do all 5 problems.

1. Let $G$ be a finite group and let $H \subseteq G$ be a subgroup of index $|G : H| = n$.
   a. Show that $|H : (H \cap H^g)| \leq n$ for all $g \in G$. (2 points)
   b. If $H$ is a maximal subgroup of $G$ and $H$ is abelian, show that $(H \cap H^g) \triangleleft G$ for all $g \notin H$. (3 points)
   c. Now suppose that $G$ is simple. If $H$ is abelian and $n$ is a prime, prove that $H = 1$. (5 points)

2. Let $K$ be a field and let $R$ be the subring of the polynomial ring $K[X]$ given by all polynomials with $X$-coefficient equal to 0.
   a. Prove that the elements $X^2$ and $X^3$ are irreducible but not prime in the ring $R$. (5 points)
   b. Show that $R$ is a Noetherian ring, and that the ideal $I$ of $R$ consisting of all polynomials in $R$ with constant term 0 is not principal. (5 points)

3. Recall that a field $K$ is algebraically closed if every polynomial $f \in K[X]$ splits over $K$ (is a product of linear factors in $K[X]$). Now let $F \subseteq E$ be an algebraic field extension.
   a. If every polynomial $f(X) \in F[X]$ splits over $E$, prove that $E$ is algebraically closed. (4 points)
   b. If every polynomial $f(X) \in F[X]$ has a root in $E$ and if $F$ has characteristic 0, prove that $E$ is algebraically closed. (6 points)

4. Let $V$ be a finite dimensional vector space over the field $F$. Suppose $T : V \rightarrow V$ is a linear operator and let $f(X) \in F[X]$ be its minimal polynomial.
   a. If $f(X)$ has a nonconstant polynomial factor of degree $m$, show that $V$ has a nonzero subspace $W$ of dimension $\leq m$ with $T(W) \subseteq W$. (5 points)
   b. Conversely, if $V$ has a nonzero subspace $W$ of dimension $n$ with $T(W) \subseteq W$, show that $f(X)$ has a nonconstant polynomial factor of degree $\leq n$. (5 points)

5. Let $R$ be a ring with 1 and let $V$ be a right $R$-module. Suppose that $V = X \oplus Y$ is the internal direct sum of the two nonzero submodules $X$ and $Y$.
   a. Show that 0, $X$, $Y$ and $V$ are the only $R$-submodules of $V$ if and only if $X$ and $Y$ are nonisomorphic simple $R$-modules. (6 points)
   b. If $X$ and $Y$ are nonisomorphic simple $R$-modules, prove that $\text{End}_R(V)$, the ring of $R$-endomorphisms of $V$, is isomorphic to the direct sum of two division rings. (4 points)
Algebra Qualifying Exam - January 2005

Do all 5 problems. Show all work.

1. Let $G$ be a finite group with $|G| = 660 = 2^2 \cdot 3 \cdot 5 \cdot 11$ and suppose that $E \subseteq G$ is a subgroup of order 11. Assume that $C_G(E) = E$.
   (a) Prove that $|N_G(E)| = 55$. (3 points)
   (b) If $M \trianglelefteq G$, show that either $E \subseteq M$ or $|M| \equiv 1 \mod 11$. (3 points)
   (c) Show that every minimal normal subgroup of $G$ contains $E$. (4 points)

2. All rings in this problem are commutative with 1. A ring $S$ is said to be finitely generated if there exist finitely many elements $s_1, s_2, \ldots, s_n \in S$ such that every element of $S$ can be written as a sum of products of these generators. Now let $R$ be a ring, let $G$ be a finite group of automorphisms of $R$, and let $R^G = \{ r \in R \mid r^g = r \text{ for all } g \in G \}$ be the fixed subring.
   (a) If $r \in R$, prove that $R^G$ contains a finitely generated subring $T$ such that $r$ is integral over $T$. (4 points)
   (b) If $R$ is finitely generated, show that $R^G$ contains a finitely generated subring $S$ such that $R$ is integral over $S$. (2 points)
   (c) Let $R$ and $S$ be as in (b). Deduce that $R$ is a finitely generated $S$-module and hence that $R^G$ is a finitely generated $S$-module. Conclude that $R^G$ is a finitely generated ring. (Hint. You can use the fact that any finitely generated ring is a homomorphic image of a polynomial ring in finitely many variables over the integers and hence is a Noetherian ring.) (4 points)

3. Let $F$ be a field and let $f(X) \in F[X]$ be an irreducible polynomial. Suppose $E \supseteq F$ is an extension field of $F$ containing a root $\alpha$ of $f(X)$ satisfying $f(\alpha^2) = 0$. Show that $f(X)$ splits over $E$. (10 points)

4. Let $F$ be an algebraically closed field and let $M_n(F)$ be the ring of $n \times n$ matrices over $F$. Describe those matrices $X \in M_n(F)$ with the property that all matrices that commute with $X$ are diagonalizable. (10 points)

5. An additive abelian group $U$ is said to be uniform if, for every two nonzero subgroups $X$ and $Y$, we have $X \cap Y \neq 0$. Let us also say that $U$ is max-uniform if $U$ is uniform and if $U$ is not contained in any properly larger uniform group.
   (a) If $U$ is uniform and has a nonzero element of finite order, show that there exists a prime $p$ such that every element of $U$ has order a power of $p$. (3 points)
   (b) Let $A$ be an abelian group and let $U$ be a uniform subgroup. Suppose $M$ is a subgroup of $A$ maximal with the property that $M \cap U = 0$. Show that $A/M$ is a uniform group. (3 points)
   (c) Let $A$ be an abelian group and let $U$ be a max-uniform subgroup. Prove that there exists a subgroup $M$ of $A$ with $A = U + M$, the internal direct sum of $U$ and $M$. Include details of the Zorn’s Lemma argument. (4 points)
1. Let $A$, $B$ and $K$ be minimal normal subgroups of the group $G$ with $K \neq A$, $K \neq B$ and $K \subseteq AB$.
   (a) Show that $KA = AB = KB$. (4 points)
   (b) Prove that $A \cong K \cong B$. (3 points)
   (c) Show that $AB$ is abelian. (3 points)

2. Let $\mathbb{Z}[x]$ be the polynomial ring over the integers $\mathbb{Z}$ in the indeterminant $x$. Let $R$ be the subring of $\mathbb{Z}[x]$ consisting of all polynomials having their coefficients of $x$ and $x^2$ equal to 0.
   (a) Prove that $\mathbb{Q}(x)$ is the field of fractions of $R$, where $\mathbb{Q}$ is the field of rational numbers. (2 points)
   (b) Find the integral closure of $R$ in $\mathbb{Q}(x)$. (4 points)
   (c) Does there exist a polynomial $g(x) \in R$ such that $R$ is generated as a ring by 1 and $g(x)$? (4 points)

3. Let $n$ be a positive integer and let $F$ be a field of characteristic not dividing $n$. Let $f(x) \in F[x]$ be the polynomial $x^n - a$ for some $0 \neq a \in F$ and let $E$ be a splitting field for $f(x)$ over $F$.
   (a) Show that $E$ contains a primitive $n$th root of unity $\varepsilon$. (3 points)
   (b) If $\varepsilon \in F$, show that all irreducible factors of $f(x)$ in $F[x]$ have the same degree and that $|E : F|$ divides $n$. (3 points)
   (c) Now assume that $n$ is a power of 2, but do not assume that $\varepsilon \in F$. Prove that $|E : F|$ is a power of 2. (4 points)

4. Let $V$ be a finite-dimensional vector space over the real numbers $\mathbb{R}$.
   (a) If $\dim_{\mathbb{R}} V$ is odd, prove that every linear operator $A : V \to V$ has at least one real eigenvalue. (3 points)
   (b) Suppose $A_1, A_2, \ldots, A_n$ are finitely many pairwise commuting linear operators on $V$. Assume that none of the operators $A_i$ has a negative real eigenvalue. If the sum $A_1 + A_2 + \cdots + A_n$ is equal to the negative of the identity operator on $V$, show that $\dim_{\mathbb{R}} V$ is even. (Hint. Use induction on the dimension of $V$.) (7 points)

5. Let $R$ be a ring with 1 and let $M$ be a right $R$-module. We say that the module $M$ has property (*) if every nonzero homomorphic image of $M$ has a simple submodule.
   (a) If $M$ is generated by its artinian submodules, show that $M$ has property (*). (5 points)
   (b) If $M$ has property (*) and is noetherian, show that it is artinian. (5 points)
Algebra Qualifying Exam
January 2007

Do all 5 problems.

1. Let $G$ be a finite group and let $\text{Syl}_p(G)$ denote its set of Sylow $p$-subgroups.
   a. Suppose that $S$ and $T$ are distinct members of $\text{Syl}_p(G)$ chosen so that $S \cap T$ is maximal among all such intersections. Prove that the normalizer $N_G(S \cap T)$ has more than one Sylow $p$-subgroup. (5 points)
   b. Show that $S \cap T = 1$ for all $S, T \in \text{Syl}_p(G)$, with $T \neq S$, if and only if $N_G(P)$ has exactly one Sylow $p$-subgroup for every nonidentity $p$-subgroup $P$ of $G$. (5 points)

2. Let $R$ be a commutative, Noetherian integral domain.
   a. If $P$ is a prime ideal of $R$, show that the radical of $P^n$ is $P$. (2 points)
   b. If $R$ has a unique nonzero prime ideal $P$, prove that all ideals of $R$ are primary. (3 points)
   c. Conversely, let us now assume that all ideals of $R$ are primary, and let $P$ and $Q$ be distinct prime ideals of $R$ with $Q \not\subseteq P$. Since $P^n \cap Q$ is primary, deduce first that $P^n \supseteq Q$ and then that $Q = 0$. (Hint. Consider whether the intersection $P^n \cap Q$ can be irredundant.) (5 points)

3. Let $F$ be a field of characteristic 0 and let $f \in F[X]$ be an irreducible polynomial of degree $> 1$ with splitting field $E \supseteq F$. Define $\Omega = \{\alpha \in E \mid f(\alpha) = 0\}$.
   a. Let $\alpha \in \Omega$ and let $m$ be a positive integer. If $g \in F[X]$ is the minimal polynomial of $\alpha^m$ over $F$, show that $\{\beta^m \mid \beta \in \Omega\}$ is the set of roots of $g$. (3 points)
   b. Now fix $\alpha \in \Omega$ and suppose that $\alpha r \in \Omega$ for some $r \in F$. Show that, for all $\beta \in \Omega$ and integers $i \geq 0$, we have $\beta r^i \in \Omega$. Conclude that $r$ is a root of unity. (3 points)
   c. If $\alpha$ and $r$ are as in (b) and if $m$ is the multiplicative order of the root of unity $r$, show that $f(X) = g(X^m)$, where $g$ is the minimal polynomial of $\alpha^m$ over $F$. (4 points)

4. Let $V$ be a finite dimensional vector space over a field $K$ and assume that $V$ is endowed with a not necessarily symmetric bilinear form $\langle \ , \ \rangle: V \times V \rightarrow K$. We let $R$ and $L$ denote the right and left radicals of $\langle \ , \ \rangle$ given by $R = \{x \in V \mid \langle V, x \rangle = 0\}$ and $L = \{x \in V \mid \langle x, V \rangle = 0\}$, so that these are both subspaces of $V$.
   a. Use the bilinear form to construct a linear transformation $T$ from $V$ to the dual space $(V/R)^*$ of $V/R$ such that $\ker(T) = L$. (6 points)
   b. Show that $\dim_K L = \dim_K R$, and deduce that the map $T$ is surjective. (4 points)

5. Let $A$ be an additive abelian group and let $B$ be a subgroup. We say that $B$ is essential in $A$, and write $B \text{ ess } A$, if and only if $B \cap X \neq 0$ for all nonzero subgroups $X$ of $A$.
   a. If $B_1 \text{ ess } A_1$ and $B_2 \text{ ess } A_2$, prove that $(B_1 \oplus B_2) \text{ ess } (A_1 \oplus A_2)$. (5 points)
   b. If $B \text{ ess } A$, and $B$ has no nonzero elements of finite order, prove that $A$ has no nonzero elements of finite order. (2 points)
   c. Let $Q$ denote the additive group of rational numbers and suppose that $Q \text{ ess } A$, for some abelian group $A$. Prove that $Q = A$. (3 points)
1. Let $G$ be a finite nonabelian group with center $Z$.
   a. If $G/Z$ is a $p$-group, for some prime $p$, show that $G$ has a normal Sylow $p$-subgroup and that $p$ divides $|Z|$. (5 points)
   b. If $G/Z$ is solvable, show that $G$ has a nonidentity normal $p$-subgroup for some prime dividing $|G : Z|$. (5 points)

2. Let $R \subseteq S$ be commutative rings with the same 1, and suppose that $S$ is finitely generated as an $R$-module.
   a. If an element $r \in R$ is not invertible in $R$, prove that it is not invertible in $S$. 
      HINT. If $r$ is invertible in $S$, consider a polynomial in $R[X]$ having $1/r$ as a root. (5 points)
   b. If the ideals of $R$ satisfy the ascending chain condition, show that the ideals of $S$ satisfy the ascending chain condition. (5 points)

3. Working in the field of complex numbers, let $\varepsilon$ be a primitive $16^{th}$ root of unity, and let $\alpha = \varepsilon\sqrt{2}$. Set $E = \mathbb{Q}[\varepsilon]$, where $\mathbb{Q}$ is the field of rational numbers, let $f(X) = X^8 + 16 \in \mathbb{Q}[X]$, and note that $\alpha$ is a root of $f(X)$.
   a. Show that $\sqrt{2} \in \mathbb{Q}[\varepsilon^2]$. (3 points)
   b. Conclude that $f(X)$ splits in $E[X]$. (2 points)
   c. If $G = \text{Gal}(E/\mathbb{Q})$, prove that no nonidentity element of $G$ fixes $\alpha$. Conclude that $f(X)$ is irreducible in $\mathbb{Q}[X]$. (5 points)

4. Let $V$ be a finite-dimensional vector space over the field $F$ of characteristic $p > 0$, let $T: V \to V$ be a linear operator on $V$, and set $W = \{v \in V \mid vT = v\}$. Suppose that $T^p = I$, the identity, and that $\dim_F W = 1$.
   a. Show that $(T - I)^p = 0$ and conclude that $\dim_F V \leq p$. (4 points)
   b. If $\dim_F V < p$, prove that $(T - I)^{p-1} = 0$. (3 points)
   c. If, for some vector $v \in V$, we have $v + vT + vT^2 + \cdots + vT^{p-1} \neq 0$, prove that $\dim_F V = p$. (3 points)

5. Let $R$ be a ring with 1. A (right) $R$-module $V$ is said to be strongly $n$-generated, for some integer $n$, if every submodule of $V$ is generated as an $R$-module by some set of $\leq n$ elements.
   a. If $V$ is strongly $n$-generated and if $W$ is a submodule of $V$, prove that both $W$ and $V/W$ are strongly $n$-generated. (3 points)
   b. Let $W$ be a submodule of $V$. If $W$ is strongly $n$-generated and if $V/W$ is strongly $m$-generated, prove that $V$ is strongly $(n + m)$-generated. (5 points)
   c. If $V$ has composition length $n$, prove that $V$ is strongly $n$-generated. (2 points)
Let $G$ be a finite group of order $p(p+1)$, where $p$ is an odd prime, and assume that $G$ does not have a normal Sylow $p$-subgroup.

(a) Find (with proof) the number of elements of $G$ with order different from $p$. (3 points)

(b) Show that each nonidentity conjugacy class of elements with order different from $p$ has size at least $p$, and conclude that there is precisely one such conjugacy class. (5 points)

(c) Prove that $p+1$ is a power of 2. (2 points)

2. Let $\mathbb{R}$ be the field of real numbers and let $\mathbb{C} \supseteq \mathbb{R}$ be the complex field. Define $S$ to be the subring of the polynomial ring $\mathbb{C}[X]$ consisting of all polynomials with real constant term so that

$$S = \mathbb{R} + \mathbb{C}X + \mathbb{C}X^2 + \mathbb{C}X^3 + \cdots.$$

(a) Show that the ideal of $S$ consisting of all polynomials with 0 constant term is not principal. (4 points)

(b) Let $I$ be a nonzero ideal of $S$ and choose $0 \neq f \in I$ to have minimal possible degree $n$. If $g \in I$, show that there exists $s \in S$ with $g - sf$ either equal to 0 or to a polynomial of degree $n$. Conclude that $I$ is generated by $f$ and perhaps one additional polynomial of degree $n$. (6 points)

3. Let $F$ be the field $\text{GF}(p)$ of prime order $p > 2$ and suppose that the polynomial $f(X) = X^m + 1 \in F[X]$ is irreducible.

(a) Show that every root of $f$ in a splitting field of the polynomial has multiplicative order $2m$. (4 points)

(b) Prove that $2m$ divides $p^m - 1$, but that $2m$ does not divide $p^n - 1$ for any integer $n$ with $0 < n < m$. (3 points)

(c) Show that $m \neq 4$. (3 points)
4. Let $V$ be a finite-dimensional vector space over the complex numbers $\mathbb{C}$ and let $T : V \to V$ be a linear operator on $V$.
(a) If $T$ is diagonalizable on $V$ and if $W$ is a subspace of $V$ with $T(W) \subseteq W$, prove that $T$ is diagonalizable on $W$. (6 points)
(b) If $T$ has the matrix
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 0
\end{pmatrix}
\]
with respect to some basis of $V$, decide (with proof) whether $T$ is diagonalizable on $V$. (4 points)

5. In the following, all groups are additive abelian groups, and recall that an abelian group is said to be noetherian if its set of subgroups satisfies the ascending chain condition or equivalently the maximal condition. Furthermore, a nonzero group is said to be uniform if it contains no direct sum of nonzero subgroups.
(a) Show that every nonzero noetherian group contains a nonzero uniform subgroup. (4 points)
(b) Suppose $G = U \oplus V$ is the internal direct sum of the two subgroups $U$ and $V$ with $U$ uniform. If $G$ contains the direct sum $A \oplus B$ with $A$ and $B$ both nonzero, prove that $(A \oplus B) \cap V \neq 0$. (3 points)
(c) Let $G = U \oplus V$ be as above with $U$ uniform. If $G$ contains the direct sum $A \oplus B \oplus C$ with $A$, $B$ and $C$ all nonzero, prove that $V$ is not uniform. (3 points)
1. Let $S_7$ denote the symmetric group on seven points, and let $A_7$ be the corresponding alternating group.
(a) Find the number of elements of order 7 in $S_7$, and find the order of the centralizer in
$S_7$ of one of these elements. (3 points)
(b) Find the order of the normalizer of a Sylow 7-subgroup in $A_7$. (3 points)
(c) Prove that $S_7$ does not contain a simple subgroup $G$ of order $504 = 2^33^27$. (4 points)

2. Let $E \supseteq K$ be fields with $|E : K| < \infty$ and let $R$ be a subring (with 1) of $K$ having $K$
as its field of fractions.
(a) Prove that there exists a ring $S$ with $R \subseteq S \subseteq E$ such that $S$ is a finitely generated
$R$-module and such that $E$ is the field of fractions of $S$. (5 points)
(b) Let $\alpha \in E$ be integral over $R$. If $R$ is integrally closed in $K$, prove that the minimal
monic polynomial $f(X) \in K[X]$ of $\alpha$ over $K$ has all its coefficients in $R$. (5 points)

3. Let $F \subseteq E$ be finite fields, where $|F| = q < \infty$ and $|E : F| = n$.
(a) Prove that every monic irreducible polynomial in $F[X]$ of degree dividing $n$ is the
minimal polynomial over $F$ of some element of $E$. (4 points)
(b) Compute the product of all the monic irreducible polynomials in $F[X]$ of degree
dividing $n$. (2 points)
(c) Suppose $|F| = 2$. Determine the number of monic irreducible polynomials of degree
10 in $F[X]$. (4 points)

4. Let $V$ be a finite dimensional vector space over the field $F$ and let $T : V \rightarrow V$ be a linear
operator on $V$ with characteristic polynomial $f(X) \in F[X]$.
(a) Show that $f(X)$ is irreducible in $F[X]$ if and only if there are no proper nonzero
subspaces $W$ of $V$ with $T(W) \subseteq W$. (6 points)
(b) If $f(X)$ is irreducible in $F[X]$ and if the characteristic of $F$ is 0, show that $T$
is diagonalizable when we extend the field $F$ to its algebraic closure. (4 points)

5. Let $R$ be a ring (with 1) and let $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = R$ be a chain of right ideals
of $R$ such that each of the $n$ quotients $V_i = I_i/I_{i-1}$ is a simple right $R$-module.
(a) If $M$ is a maximal right ideal of $R$, prove that $R/M$ is isomorphic as a right $R$-module
to some $V_i$. (3 points)
(b) Now assume that the $V_i$'s are pairwise nonisomorphic $R$-modules. Prove that the
intersection of all the maximal right ideals of $R$ is equal to 0. (5 points)
(c) Continue to assume that the $V_i$'s are pairwise nonisomorphic $R$-modules and deduce
that $R$ is a finite ring direct sum of division rings. (2 points)
1. Let the group $G = H \times K$ be the (internal) direct product of its subgroups $H$ and $K$. Suppose that there exists a group $X$ and surjective homomorphisms $\theta: H \to X$ and $\phi: K \to X$. Then we let

$$U = \{ hk \in G \mid h \in H, k \in K, \text{ and } \theta(h) = \phi(k) \}.$$ 

(a) Show that $U$ is a subgroup of $G$ such that $UH = G = UK$, $U \cap H = \ker \theta$ and $U \cap K = \ker \phi$. (5 points)
(b) If $V$ is a subgroup of $G$ with $V \supseteq U$, show that both $V \cap H$ and $V \cap K$ are normal subgroups of $G$. (2 points)
(c) If $X$ is a simple group, prove that $U$ is a maximal subgroup of $G$ that contains neither $H$ nor $K$. (3 points)

2. Let $R$ be a commutative ring with 1, let $(a) = aR$ be the principal ideal of $R$ generated by $a \in R$ and let $P$ be a prime ideal of $R$ properly contained in $(a)$.

(a) Show that $P$ is a subgroup of $G$ such that $UH = G = UK$, $U \cap H = \ker \theta$ and $U \cap K = \ker \phi$. (4 points)
(b) Now assume that $P$ is a finitely generated ideal and prove that there exists $b \in R$ with $(1 - ab)P = 0$. (Hint. Here $1 - ab$ is the determinant of an appropriate matrix with coefficients in $R$.) (5 points)
(c) In particular, if $R$ is a domain conclude that either $P = 0$ or $(a) = R$. (1 point)

3. Let $F \subseteq E$ be fields of characteristic 0, and suppose that $E = F[\alpha]$, where $\alpha^p \in F$ for some prime $p$. Now let $E^* = E[\varepsilon]$ where $\varepsilon$ is a primitive $p$-th root of 1.

(a) Show that $E^*$ is a Galois extension of $F$. (3 points)
(b) If $E$ is Galois over $F$, prove that $E = F$ or $E = E^*$. (5 points)
(c) Show, by example, that it is possible to have $E = E^*$ even if $F$ does not contain $\varepsilon$. (2 points)
4. Let $V$ be a finite-dimensional vector space over the complex numbers $\mathbb{C}$ and let $T: V \to V$ be a linear operator on $V$.

(a) Suppose $W$ is a subspace of $V$ with $T(W) \subseteq W$. Then the restriction $S$ of $T$ to $W$ is a linear operator on $W$. Prove that the characteristic polynomial $f_S(x)$ of $S$ (on $W$) divides the characteristic polynomial $f_T(x)$ of $T$ (on $V$). (4 points)

(b) Let $\lambda$ be a root of $f_T(x)$ of multiplicity $m$ and let $V_\lambda = \{ v \in V \mid T(v) = \lambda v \}$, so that $V_\lambda$ is a subspace of $V$. Prove that $1 \leq \dim_{\mathbb{C}} V_\lambda \leq m$. (4 points)

(c) Find an example of a vector space $V$, a linear operator $T$ and a root $\lambda$ of $f_T(x)$ such that $\lambda$ has multiplicity 5 as a root of the polynomial, but $\dim_{\mathbb{C}} V_\lambda = 1$. (2 points)

5. Let $R$ be a (noncommutative) ring with 1 and suppose that $R$ has a unique maximal right ideal $M$.

(a) Show that $M$ is a two-sided ideal of $R$. (3 points)

(b) Prove that every element of $R \setminus M$ has a two-sided inverse. (5 points)

(c) Show that 0 and 1 are the only idempotent elements of $R$. (2 points)
Algebra Qualifying Exam
January 2012

Do all 5 problems.

1. Let $G$ be a finite group of order $4312 = 2^3 \cdot 7 \cdot 11$.
   (a) Show that $G$ has a subgroup of order 77. (4 points)
   (b) Prove that $G$ has a subgroup of order 7 whose normalizer in $G$ has index dividing 8. (4 points)
   (c) Conclude that $G$ is not simple. (2 points)

2. Let $R$ be a commutative ring with 1, and let $Q$ be a primary ideal of $R$. Suppose that
   
   $Q = \bigcap_{i=1}^{k} X_i$

   is a finite intersection of the ideals $X_i$.
   (a) If each $X_i$ is a prime ideal, prove that $Q = X_j$ for some $j$. (Hint. You might wish to first prove that $Q$ is prime.) (5 points)
   (b) Now suppose that $R$ is Noetherian and that each $X_i$ is primary. Assume also that the radicals of the $X_i$ are distinct. Again show that $Q = X_j$ for some $j$. (5 points)

3. Let $K \subseteq F \subseteq E$ be fields with $|E : F| < \infty$, and let $A$ be the subfield of $E$ consisting of all elements of $E$ that are algebraic over $K$. Assume that $F \cap A = K$.
   (a) If $\alpha \in A$ and $f(x)$ is the monic minimal polynomial of $\alpha$ over $F$, show that all coefficients of $f(x)$ lie in $K$. (4 points)
   (b) Now assume that $K$ has characteristic 0. If $B$ is an intermediate field with $K \subseteq B \subseteq A$ and $|B : K| < \infty$, prove that $|B : K| \leq |E : F|$. (4 points)
   (c) Conclude that $|A : K| \leq |E : F|$. (2 points)

4. Let $V$ be a nonzero finite-dimensional vector space over the complex numbers.
   (a) If $S$ and $T$ are commuting linear operators on $V$, prove that each eigenspace of $S$ is mapped into itself by $T$. (2 points)
   (b) Now let $A_1, A_2, \ldots, A_k$ be finitely many linear operators on $V$ that commute pairwise. Prove that they have a common eigenvector in $V$. (4 points)
   (c) If $V$ has dimension $n$, show that there exists a nested sequence of subspaces

   $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$

   where each $V_j$ has dimension $j$ and is mapped into itself by each of the operators $A_1, A_2, \ldots, A_k$. (4 points)

5. Let $K$ be a field and assume that $-1$ is not a square in $K$. Let $G = GL(2, K)$ be the group of invertible $2 \times 2$ matrices over $K$.
   (a) If $g \in G$, show that $g$ has order 4 if and only if $\det(g) = 1$ and $\text{tr}(g) = 0$. (5 points)
   (b) Find explicitly an element $g \in G$ of order 4. (1 point)
   (c) Suppose there exist elements $a, b \in K$ with $a^2 + b^2 = -1$. Show that $G$ contains two elements $g, h$ of order 4 such that the product $gh$ also has order 4. (4 points)
ALGEBRA QUALIFYING EXAM, JANUARY 2013

In all cases, when an example is requested, you should both provide the example and proof that the object you write down actually is an example.

(1) A finite group $G$ is said to have property C if, whenever $g \in G$ and $n$ is an integer relatively prime to the order of $G$, $g$ and $g^n$ are conjugate in $G$.

(a) (3pt) Give infinitely many non-isomorphic finite groups which have property C.

(b) (3pt) Give infinitely many non-isomorphic finite groups which do not have property C.

(c) (4pt) Show that if $G$ has property C and $\rho : G \to GL_m(\mathbb{C})$ is a homomorphism, then the trace of $\rho(g)$ lies in $\mathbb{Q}$ for every $g \in G$.

(2) Let $k$ be a field. We say a polynomial $f$ in $k[x]$ is a consecutive-root polynomial if it has two roots $x_0, x_1$ (not necessarily in $k$) which satisfy $x_1 - x_0 = 1$.

(a) (2pt) Show there is no irreducible consecutive-root polynomial in $\mathbb{Q}[x]$.

(b) (3pt) Let $p$ be a prime number. Show that the polynomial $x^p - x - 1$ in $\mathbb{F}_p[x]$ is irreducible and consecutive-root.

(c) (5pt) Describe the set of irreducible monic consecutive-root polynomials in $\mathbb{F}_p[x]$ of degree at most $p$.

(3) We say a ring $R$ is von Neumann regular if, for every $a \in R$, there exists an $x \in R$ such that $a = axa$. The element $x$ is called a weak inverse of $a$. In particular, every division algebra is von Neumann regular (just take $x = 0$ if $a = 0$ and $x = a^{-1}$ otherwise.)

(a) (4pt) Give an example of a commutative von Neumann regular ring which is not a field.

(b) (2pt) Let $R$ be $M_2(\mathbb{C})$ and let $a$ be the nilpotent matrix $e_{12}$ which sends $e_1$ to 0 and $e_2$ to $e_1$. Give a weak inverse for $a$.

(c) (4pt) Prove that if $V$ is a vector space over a field $k$, the ring of endomorphisms $End_k V$ is von Neumann regular.

(4) Recall that a right module $P$ for a ring $R$ is said to be projective if, for every surjection of right $R$-modules $f : N \to P$, there is a map $g : P \to N$ such that $g$ followed by $f$ is the identity on $P$.

(a) (3pt) Prove that a free $R$-module is projective.

(b) (3pt) Prove that a right $R$-module $M$ is projective if and only if there is another right module $N$ such that $M \oplus N$ is a free right $R$-module.

(c) (4pt) In linear algebra, a “projection” is a matrix $A$ such that $A^2 = A$. More generally, if $R$ is a commutative ring, we might say that an $R$-projection is an $R$-module homomorphism $A : R^n \to R^n$ such that $A^2 = A$. For $R$ a commutative ring, prove that a finitely generated $R$-module $M$ is projective if and only if it is isomorphic to the image of some projection.
(5) Let $W_n$ be the set of $n \times n$ complex matrices $C$ such that the equation

$$AB - BA = C$$

has a solution in $n \times n$ matrices $A, B$.

(a) (2pt) Show that $W_n$ is closed under scalar multiplication and conjugation.

(b) (4pt) Show that the identity matrix is not in $W_n$.

(c) (4pt) Give a complete description of $W_2$ (i.e. a criterion for determining whether a matrix $C$ is in $W_2$ other than “look around for matrices $A$ and $B$ such that $AB - BA = C$.”)
Algebra Qualifying Exam
September 1991

Do all 5 problems.

1. Let $p$ and $q$ be distinct primes and suppose $G$ is a finite group having precisely $p + 1$ Sylow $p$-subgroups and $q + 1$ Sylow $q$-subgroups. Prove that there exist $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ such that the subgroup of $G$ generated by $P$ and $Q$ is $PQ = P \times Q$.

2. Let $R$ be a commutative ring with 1. If $a \in R$, we write $\text{ann}(a) = \{ r \in R \mid ar = 0 \}$ for the annihilator of $a$ in $R$. Thus $\text{ann}(a)$ is an ideal of $R$ and we let $S \subseteq R$ be the set of all elements $a \in R$ such that $\text{ann}(a)$ is a prime ideal of $R$.
   i. (4 points) If $R$ is Noetherian, show that $S$ is nonempty.
   ii. (4 points) If $a \in S$ and $r \in R$, show that either $ar = 0$ or $ar \in S$.
   iii. (2 points) If $a, b \in S$ and $\text{ann}(a) \neq \text{ann}(b)$, prove that $ab = 0$.

3. Let $F \subseteq E$ be an algebraic extension of fields. We say that an element $\alpha$ of $E$ is abelian if $F[\alpha]$ is a Galois extension of $F$ with abelian Galois group $\text{Gal}(F[\alpha]/F)$. Prove that the set of abelian elements of $E$ is a subfield of $E$ containing $F$.

4. Let $V$ be a finite-dimensional vector space over a field $K$ and let $(\ , \ )$ be a bilinear form on $V$. Suppose $T: V \to V$ is a linear transformation satisfying $(v, Tw) = (Tv, w)$ for all $v, w \in V$. Write $N = \ker(T) = \{ v \in V \mid Tv = 0 \}$.
   i. (5 points) Assume that the form restricted to $N$ is nondegenerate – that is, if $v \in N$ and $(v, N) = 0$, then $v = 0$. If $T$ is nilpotent, prove that $T = 0$.
   ii. (5 points) Find a 2-dimensional example with $T$ a nonzero nilpotent transformation and with the form $(\ , \ )$ nondegenerate on the whole vector space $V$.

5. Let $R$ be a ring with 1 and let $M$ be a right $R$-module. Suppose

\[ 0 = M_0 \subset M_1 \subset \cdots \subset M_n = M \]

is a chain of submodules such that, for $i = 1, 2, \ldots, n$, the factors $M_i/M_{i-1}$ are simple and pairwise nonisomorphic. If $X$ and $Y$ are isomorphic submodules of $M$, prove that $X = Y$. 
Algebra Qualifying Exam
August 1992

Do all 5 problems.

1. Let \( x \) and \( y \) be elements of a finite \( p \)-group \( P \) and let \( z = [x,y] \) be the commutator \( x^{-1}y^{-1}xy \) of \( x \) and \( y \). Suppose that \( x \) lies in every normal subgroup of \( P \) which contains \( z \). Prove that \( x = 1 \).

2. Let \( K[x] \) be a polynomial ring over the field \( K \) and let \( R \) be the subring of \( K[x] \) consisting of all polynomials whose \( x \)-coefficient is equal to 0. Thus a typical element of \( R \) has the form

\[ a_0 + a_2x^2 + a_3x^3 + \cdots + a_nx^n \]

with \( a_i \in K \). Show that the principal ideal \( (x^2) = x^2R \) is a primary ideal of \( R \) which is not equal to a power of its radical.

3. Let \( E \) be a finite degree field extension of the rationals \( \mathbb{Q} \) and suppose that \( f(x) \) is a monic irreducible polynomial in \( E[x] \).
   i. Show that there exists a unique monic irreducible polynomial \( g(x) \in \mathbb{Q}[x] \) such that \( f(x) \) divides \( g(x) \) in \( E[x] \). (4 points)
   ii. Let \( g(x) \) be as above. If \( E \) is a splitting field over \( \mathbb{Q} \) for some polynomial in \( \mathbb{Q}[x] \), show that the degree of \( f(x) \) divides the degree of \( g(x) \). (4 points)
   iii. Give an example to show that the degree of \( f(x) \) need not divide the degree of \( g(x) \) in general. (2 points)

4. Let \( V \) be a finite dimensional vector space over a field \( K \) and let \( B: V \times V \rightarrow K \) be a bilinear form. Suppose that for all \( x, y \in V \) we have \( B(x,y) = 0 \) if and only if \( B(y,x) = 0 \).
   i. If \( v, w \in V \) with \( B(v,v) \neq 0 \), prove that \( B(v,w) = B(w,v) \). (5 points)
   ii. Deduce that either \( B \) is symmetric or \( B(v,v) = 0 \) for all \( v \in V \). (5 points)

5. Let \( G \) be the multiplicative group of all \( 2 \times 2 \) matrices over the integers \( \mathbb{Z} \) whose determinant is equal to 1. Notice that \( G \) acts by right multiplication on the set \( \Omega \) of all 1-dimensional subspaces of the 2-dimensional row space \( \mathbb{Q}^2 \) over the rational numbers \( \mathbb{Q} \).
   i. Find all elements of \( G \) which act trivially, that is which fix every element of \( \Omega \). (4 points)
   ii. Prove that \( G \) acts transitively. In other words, show that \( \Omega \) is an orbit under the action of \( G \). (6 points)
Algebra Qualifying Exam
August 1993

Do all 5 problems.

1. Let $G$ be a group of order $|G| = 504 = 7 \cdot 8 \cdot 9$.
   i. If there exists an element $x \in G$ of order 21, show that there exists a subgroup $H < G$ having index $|G : H| = 8$. Hint. Find the possibilities for $n_7$, the number of Sylow 7-subgroups of $G$. (6 points)
   ii. If $G$ is simple, show that it contains no element of order 21. (4 points)

2. Let $R$ be a ring and let $M$ be a right $R$-module which has a composition series. Assume that $M$ has a unique simple submodule $N$ and that $N$ is not isomorphic to any composition factor of $M/N$. Prove that the ring $\text{End}_R(M)$ is a division ring. In other words, prove that every nonzero $R$-endomorphism of $M$ is one-to-one and onto.

3. Let $K \subseteq E$ be fields and let $f(x) \in K[x]$ be a polynomial of degree $n \geq 2$ having $n$ distinct roots $a_1, a_2, \ldots, a_n$ in $E$. Suppose that the field extension $K[a_1, a_2]/K$ has degree $|K[a_1, a_2] : K| = n(n-1)$.
   i. Find the degrees of the irreducible factors of $f(x)$ over the field $K$ and over the field $K[a_1]$. (3 points)
   ii. If $g(x)$ is the minimal polynomial of $a_1 + a_2$ over $K$, prove that $a_i + a_j$ is a root of $g(x)$ for all $i \neq j$. Hint. First consider the case $i = 1$. (7 points)

4. Let $A$ be an $m \times n$ matrix over the integers $\mathbb{Z}$ and consider the system of homogeneous linear equations $AX = 0$ where $X$ is the column vector of unknowns $x_1, x_2, \ldots, x_n$. Suppose that every integer solution of this system has all $x_i$'s equal. Prove that the same is true for every real solution of this system of equations.

5. Let $K = \mathbb{Q}[i]$ be the field generated over the rationals $\mathbb{Q}$ by $i = \sqrt{-1}$ and let $R$ be the subring of $K$ defined by

   \[ R = \{ a + bi \mid a, b \in \mathbb{Z} \} \]

   where $\mathbb{Z}$ is the ring of integers. Suppose $\alpha$ is a complex number which is the root of a monic polynomial in $R[x]$. Prove that the minimal monic polynomial of $\alpha$ over $K$ has all coefficients in $R$. 
Algebra Qualifying Exam
August 1994

Do all 5 problems.

1. Let $G$ be a finite group and let $P \in \text{Syl}_p(G)$ for some prime $p$. Suppose $N$ is a normal subgroup of $G$ with $|G : N| = |P| > 1$.
   i. Prove that $N$ is the subset of $G$ consisting of all elements of order not divisible by $p$. (4 points)
   ii. If the elements of $G$ outside of $N$ all have $p$-power order, prove that $P$ is its own normalizer. (6 points)

2. Let $R$ be a commutative ring and let $P$ be a prime ideal of $R$. If $V$ is a (right) $R$-module, define
   $$W = \{ v \in V \mid va = 0 \text{ for some } a \in R \setminus P \}$$
   where $R \setminus P$ is the set of elements of $R$ not in $P$.
   i. Show that $W$ is an $R$-submodule of $V$. (2 points)
   ii. If $R$ is Noetherian and $V$ is a finitely generated $R$-module, prove that $Wb = 0$ for some $b \in R \setminus P$. (3 points)
   iii. If $V$ is a simple $R$-module and $W = 0$, prove that $P$ is a maximal ideal. (5 points)

3. Let $E$ be a splitting field of the polynomial $x^3 - 2$ over the rationals $\mathbb{Q}$, and assume that $E$ is contained in the complex numbers $\mathbb{C}$. Let $F = E \cap \mathbb{R}$ be the real subfield of $E$, and note that $F = \mathbb{Q}[\sqrt[3]{2}]$.
   i. Show that $G = \text{Gal}(E/\mathbb{Q})$ contains an element $\sigma$ with the property that the only elements of $F$ fixed by $\sigma$ are rational. (4 points)
   ii. Let $a \in F$ and suppose that $a^3 \in \mathbb{Q}$. Show that one of $a$, $a \sqrt[3]{2}$, or $a \sqrt[3]{4}$ is contained in $\mathbb{Q}$. (4 points)
   iii. Prove that $\sqrt[3]{3} \notin E$. (2 points)

4. Let $A$ be an $n \times n$ matrix over a field $K$ and assume that the characteristic polynomial of $A$ has distinct roots in the algebraic closure of $K$. Prove that any two $n \times n K$-matrices which commute with $A$ must commute with each other.

5. Let $S = M_n(F)$ be the ring of $n \times n$ matrices over the field $F$.
   i. If $s \in S$ is nilpotent, prove that the trace of $s$ is zero. (4 points)
   ii. Suppose $R$ is a ring and that $\theta: R \to S$ is a surjective ring homomorphism. Let $I$ be an ideal of $R$ with the property that every element of $I$ is a sum of nilpotent elements of $R$. Show that $\theta(I) = 0$. (6 points)
1. Let $G$ be a finite group. We say that a subgroup $M$ of $G$ has property $(\ast)$ if $M$ is abelian, maximal, and not normal in $G$.
   i. If $M$ and $N$ are distinct subgroups of $G$ with property $(\ast)$, prove that $M \cap N = Z$, where $Z = Z(G)$ is the center of $G$. (2 points)
   ii. Let $M$ have property $(\ast)$ and let $S(M)$ denote the set of all noncentral elements of $G$ which are conjugate to elements of $M$. Note that 
   \[ S(M) = \bigcup_{x \in G} (M \setminus Z)^x. \]
   Compute the cardinality $|S(M)|$ of $S(M)$ in terms of $|M| = m$, $|Z| = z$, and $|G| = g$. Deduce that $g - z > |S(M)| > (g - z)/2$. (5 points)
   iii. Show that any two subgroups of $G$ having property $(\ast)$ must be conjugate in $G$. (3 points)

2. Let $R$ be a ring. If $V$ and $W$ are right $R$-modules, we write $V \sim W$ when $V$ is isomorphic to a submodule of $W$ and $W$ is isomorphic to a submodule of $V$.
   i. If $V \sim W$ and if $V$ satisfies the minimum condition, prove that $V$ and $W$ are isomorphic. (4 points)
   ii. Suppose $R = \mathbb{Z}$ is the ring of integers. If $V \sim W$ and if $V$ is finitely generated, prove that $V$ and $W$ are isomorphic. (3 points)
   iii. Suppose $R$ is a commutative integral domain and let $I$ be a nonzero ideal of $R$. Show that $R \sim I$ when we view $R$ and $I$ as right $R$-modules. Conclude that if $R$ is not a PID, then there exist nonisomorphic $R$-modules $V$ and $W$ with $V \sim W$. (3 points)

3. Let $E$ be the subfield of the real numbers generated over $\mathbb{Q}$ by $\sqrt{2}$ and $\sqrt[3]{2}$.
   i. Show that $|E : \mathbb{Q}| = 6$. (2 points)
   ii. If $K$ is a field with $\mathbb{Q} \subseteq K \subseteq E$, show that $K$ is one of the fields $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt[3]{2}]$, or $E$. (5 points)
   iii. Prove that $E = \mathbb{Q}[\sqrt{2} + \sqrt[3]{2}]$. (3 points)
4. Let $V$ and $W$ be finite-dimensional vector spaces over an algebraically closed field $F$ and let $A: V \rightarrow V$ and $B: W \rightarrow W$ be linear operators. Suppose $T: V \rightarrow W$ is a nonzero linear transformation such that $T(A(v)) = B(T(v))$ for all $v \in V$, and let $N = \ker T$.
   i. Show that $A(N) \subseteq N$. (2 points)
   ii. Show that there exists $\lambda \in F$ and a vector $v \in V$ with $v \notin N$ such that $A(v) - \lambda v \in N$. (4 points)
   iii. If $\lambda$ is as in part (ii), show that $\lambda$ is an eigenvalue for both $A$ and $B$. (4 points)

5. Let $S$ be the set of all $2 \times 2$ complex matrices of the form
   \[
   \begin{bmatrix}
   a & \bar{b} \\
   b & \bar{a}
   \end{bmatrix}
   \]
   with $a, b \in \mathbb{C}$ and where, as usual, $\bar{\cdot}$ denotes complex conjugation.
   i. Show that $S$ is a subring of the ring $M_2(\mathbb{C})$ of all $2 \times 2$ matrices over $\mathbb{C}$. (2 points)
   ii. Determine the center $Z$ of $S$ and show that $Z$ is isomorphic to the real numbers $\mathbb{R}$. (3 points)
   iii. Prove that
   \[
   I = \left\{ \begin{bmatrix} x & \bar{x} \\ x & \bar{x} \end{bmatrix} \bigg| x \in \mathbb{C} \right\}
   \]
   is a minimal right ideal of $S$ and that it is faithful as a right $S$-module. (3 points)
   iv. Show that $\dim \mathbb{Z} I = 2$ and conclude that $S \cong M_2(\mathbb{R})$. (2 points)
Algebra Qualifying Exam
August 1996

Do all 5 problems.

1. We say that a group $G$ has property ($\ast$) if every normal abelian subgroup of $G$ is contained in $Z(G)$, the center of $G$.
   a. Suppose that $N$ and $M$ are normal subgroups of a group $G$ and that $G/N$ and $G/M$ have property ($\ast$). Prove that $G/(N \cap M)$ has property ($\ast$). (3 points)
   b. Let $N \triangleleft G$ and assume that $G/N$ has property ($\ast$). If $N$ has no nontrivial abelian normal subgroups, prove that $G$ has property ($\ast$). (3 points)
   c. Show that a finite $p$-group with property ($\ast$) must be abelian. (4 points)

2. Let $R$ be a commutative ring with 1, let $n \geq 2$ be a fixed integer, and suppose that $x^n = x$ for all $x \in R$.
   a. If $P$ is a prime ideal of $R$, show that $R/P$ is a finite field containing at most $n$ elements. (4 points)
   b. Prove that the intersection of all prime ideals of $R$ is the zero ideal. (2 points)
   c. If $R$ is a Noetherian ring, conclude that $R$ is finite. (4 points)

3. Let $f(x) = x^6 + 3 \in \mathbb{Q}[x]$, let $\alpha$ be a root of $f(x)$ in $\mathbb{C}$, and set $E = \mathbb{Q}[\alpha]$.
   a. Show that $E$ contains a primitive 6th root of unity. (3 points)
   b. Prove that $E$ is Galois over $\mathbb{Q}$. (2 points)
   c. Count the number of intermediate fields $F$ with $\mathbb{Q} \subseteq F \subseteq E$ and $|F : \mathbb{Q}| = 3$. Justify your answer. (5 points)

4. Let $V$ be a finite dimensional vector space over a field $K$ and let $T : V \to V$ be a linear operator. Assume that there exists a nonzero vector $v \in V$ such that $V$ is spanned by the vectors $vT^i$ for $i = 0, 1, 2, \ldots$.
   a. Show that there exists a basis for $V$ with respect to which the matrix of $T$ has the form
      \[
      \begin{bmatrix}
      0 & 1 \\
      0 & 1 & 1 \\
      \vdots & \ddots & \ddots \\
      a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & 0 & 1
      \end{bmatrix}
      \]
      for suitable $a_i \in K$. (5 points)
   b. Prove that the minimal polynomial and the characteristic polynomial of $T$ are identical. (5 points)
5. The goal of this problem is to prove:

**Theorem.** Let $M_3(F)$ denote the space of $3 \times 3$ matrices over the field $F$. The following are equivalent.

i. $F$ has an extension field of degree 3.
ii. $M_3(F)$ contains a 3-dimensional subspace whose nonzero members are all invertible matrices.
iii. $M_3(F)$ contains a 2-dimensional subspace whose nonzero members are all invertible matrices.

a. If $E$ is an extension field of $F$ of degree 3, show that the ring $M_3(F)$ contains an isomorphic copy of $E$. Deduce that (i) implies (ii). (5 points)

b. Let $A$ and $B$ be linearly independent invertible matrices in $M_3(F)$. If the characteristic polynomial of $AB^{-1}$ is not irreducible over $F$, show that some nonzero $F$-linear combination of $A$ and $B$ is not invertible. Deduce that (iii) implies (i). (5 points)
Algebra Qualifying Exam
August 1997

Do all 5 problems.

1. Suppose that $G$ is a finite group that has exactly 50 Sylow 7-subgroups. Let $P \in \text{Syl}_7(G)$ and write $N = \mathbb{N}_G(P)$.
   a. Show that $N$ is a maximal subgroup of $G$. (5 points)
   b. If $N$ has a normal Sylow 5-subgroup $Q$, prove that $Q \triangleleft G$. (5 points)

2. Let $R$ be a commutative domain with 1.
   a. Let $a, b \in R$ and assume that the principal ideal $(ab)$ is primary. If $a \neq 0$ and $b$ is not a unit of $R$, prove that $b^n \in (a)$ for some integer $n \geq 1$. (4 points)
   b. Now assume that every principal ideal of $R$ is primary and let $P$ be any nonzero prime ideal of $R$. Show that $P$ contains every nonunit of $R$. Deduce that $P$ is the unique nonzero prime ideal of $R$. (6 points)

3. Let $\alpha$ be a nonzero real number and suppose that $\alpha^n \in \mathbb{Q}$, the rational numbers, for some positive integer $n$. Let $g(x)$ be the minimal (monic) polynomial of $\alpha$ over $\mathbb{Q}$, and suppose that $\deg g = m$.
   a. Show that $g(0) = \pm \alpha^m$. (5 points)
   b. Deduce that $g(x) = x^m - b$ for some rational number $b$. (2 points)
   c. Prove that $m$ divides $n$. (3 points)

4. Let $V$ be a finite dimensional vector space over an algebraically closed field $K$ and let $T : V \to V$ be a linear operator. Also let $I : V \to V$ denote the identity operator. Show that $V$ has a basis consisting of eigenvectors of $T$ if and only if the kernel of $(\lambda I - T)^2$ is equal to the kernel of $\lambda I - T$ for all choices of $\lambda \in K$. (5 points for each direction)

5. Let $R$ be a ring with 1 and let $V$ be a right $R$-module. Suppose $X$ and $Y$ are $R$-submodules of $V$ such that $V = X \oplus Y$, the (internal) direct sum. If $\theta$ is any $R$-homomorphism from $X$ to $Y$, define $W_\theta \subseteq V$ to be the set of elements $x - \theta(x) \in V$ for all $x \in X$.
   a. Show that $W_\theta$ is an $R$-submodule of $V$ and that $V = W_\theta \oplus Y$. (4 points)
   b. Conversely, suppose $U$ is an $R$-submodule of $V$ such that $V = U \oplus Y$. Prove that $U = W_\theta$ for some $R$-homomorphism $\theta : X \to Y$. (6 points)
Algebra Qualifying Exam
August 1998

Do all 5 problems.

1. If $G$ is a finite group, we define $\text{soc}(G)$ to be the subgroup generated by all the minimal normal subgroups of $G$.
   a. If $(1) \neq N \triangleleft G$, show that $N \cap \text{soc}(G) \neq (1)$. (2 points)
   b. Prove that $\text{soc}(\text{soc}(G)) = \text{soc}(G)$. (4 points)
   c. If $\text{soc}(G) = G$, show that every minimal normal subgroup of $G$ is simple. (4 points)

2. Let $R$ be a commutative domain with 1 and let $F$ be its field of fractions. For any element $q \in F$, we define $I_q = \{ r \in R \mid rq \in R \}$.
   a. Show that each $I_q$ is a nonzero ideal of $R$. (2 points)
   b. If $M$ is a maximal ideal of $R$, let $R_M = \{ a/b \in F \mid a \in R, b \in R, b \notin M \}$ and recall that $R_M$ is a subring of $F$ that contains $R$. Prove that $R = \bigcap_M R_M$, where the latter intersection is over all maximal ideals $M$ of $R$. (5 points)
   c. Now suppose that $R = \mathbb{Z}[\sqrt{-3}]$ and let $q = (1 - \sqrt{-3})/2 = 2/(1 + \sqrt{-3}) \in F$. Show that $I_q$ is not a principal ideal. (Hint. Use the fact that the norm map $N(\alpha) = |\alpha|^2$ is multiplicative.) (3 points)

3. Let $F$ be a field and suppose $f(x) \in F[x]$ is an irreducible polynomial. Fix an integer $n$ and let $g(x) = f(x^n)$.
   a. If $h(x)$ is any irreducible factor of $g(x)$ in $F[x]$, show that the degree of $h(x)$ is a multiple of the degree of $f(x)$. (5 points)
   b. Now suppose that $F$ has characteristic 0 and that it contains a primitive $n$th root of unity. Show that all irreducible factors of $g(x)$ in $F[x]$ have equal degrees. (5 points)

4. Let $V$ be a finite dimensional vector space over the field $K$ and let $(\ ,\ ): V \times V \to K$ be a symmetric bilinear form. For any subspace $U$ of $V$, we let $U^\perp = \{ v \in V \mid (U, v) = 0 \}$. Thus $U^\perp$ is also a subspace of $V$, and the form is nonsingular precisely when $V^\perp = 0$.
   a. Show that $\dim U + \dim U^\perp \geq \dim V$ for any subspace $U$ of $V$. (4 points)
   b. If the form is nonsingular and if $V = U + X$ is the sum of subspaces $U$ and $X$, prove that $\dim U^\perp + \dim X^\perp \leq \dim V$. (2 points)
   c. If the form is nonsingular, show that $\dim U + \dim U^\perp = \dim V$ for any subspace $U$ of $V$. (4 points)

5. Let $A$ be a (not necessarily finite) abelian group and let $B$ be a subgroup of $A$.
   a. If $B$ is a direct factor of $A$, show that $B$ is a direct factor of every subgroup $C$ satisfying $B \subseteq C \subseteq A$. (4 points)
   b. Conversely, assume that $B$ is a direct factor of every subgroup $C$ such that $B \subseteq C \subseteq A$ and $C/B$ is cyclic. If $|A : B| < \infty$, show that $B$ is a direct factor of $A$. (6 points)
Algebra Qualifying Exam
August 1999

Do all 5 problems.

1. Let $G$ be a group and let $K \subseteq H$ be subgroups of $G$ with $K \trianglelefteq H$.
   a. Prove that $H$ normalizes $C_G(K)$. (3 points)
   b. If $H \trianglelefteq G$ and $C_H(K) = \langle 1 \rangle$, prove that $H$ centralizes $C_G(K)$. (7 points)

2. In this problem, the word ideal always means two-sided ideal. Now let $R$ be a (not necessarily commutative) ring with 1. An ideal $P$ of $R$ is said to be prime if, for all ideals $A$ and $B$ of $R$, the inclusion $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.
   a. If an ideal $Q$ is not prime, show that there exist ideals $A > Q$ and $B > Q$ with $AB \subseteq Q$. (2 points)
   b. Let $I$ and $Q$ be ideals of $R$ and assume that $Q$ is maximal with the property that it contains no power of $I$. Show that $Q$ is prime. (4 points)
   c. Suppose $I$ is a nonnilpotent ideal of $R$. If $R$ satisfies the ascending chain condition on ideals, prove that there exists a prime ideal of $R$ which does not contain $I$. (4 points)

3. Let $K \subseteq L$ be a finite degree extension of fields. Suppose that $E$ and $F$ are intermediate fields, each Galois over $K$, and that $L = EF$ is the field generated by $E$ and $F$. (This means that no proper subfield of $L$ contains both $E$ and $F$.)
   a. Prove that $L$ is Galois over $K$. (4 points)
   b. If $\text{Gal}(E/K) = G$ and $\text{Gal}(F/K) = H$, show that $\text{Gal}(L/K)$ is isomorphic to a subgroup of $G \times H$. (6 points)

4. In this problem, all matrices are viewed over the complex numbers.
   a. For which complex numbers $x$, if any, is the matrix $\begin{bmatrix} 1 & -2 \\ 8 & x \end{bmatrix}$ not similar to a diagonal matrix? Explain. (5 points)
   b. Let $J$ be the $n \times n$ matrix all of whose entries are equal to 1. Find a diagonal matrix similar to $J$ or prove that none exists. (5 points)

5. Let $F[x, y]$ be the polynomial ring over the field $F$ in the two indeterminates $x$ and $y$. Suppose $f(x) \in F[x] \subseteq F[x, y]$ and $g(y) \in F[y] \subseteq F[x, y]$ are polynomials of positive degree in the indeterminates $x$ and $y$, respectively. Let $I = (f(x), g(y))$ be the ideal of $F[x, y]$ generated by $f(x)$ and $g(y)$.
   a. Prove that $I \neq F[x, y]$. (5 points)
   b. If $f(x) = x - \alpha$ and $g(y) = y - \beta$ for some $\alpha, \beta \in F$, show that $I$ is a maximal ideal of $F[x, y]$. (5 points)
Algebra Qualifying Exam
August 2000

Do all 5 problems.

1. Suppose that a group $G$ is the (internal) direct product of subgroups $S$ and $T$. Let $H$ be a subgroup of $G$ such that $SH = G = TH$.
   a. Prove that $S \cap H$ and $T \cap H$ are normal subgroups of $G$. (4 points)
   b. If $S \cap H = 1 = T \cap H$, prove that $S$ and $T$ are isomorphic. (3 points)
   c. If $S \cap H = 1 = T \cap H$ and $H$ is normal in $G$, show that $G$ is abelian. (3 points)

2. Let $A_1, A_2, \ldots, A_n$ be ideals of the commutative ring $R$, and let $D = \bigcap_{i=1}^{n} A_i$.
   a. Prove that $\sqrt{D} = \bigcap_{i=1}^{n} \sqrt{A_i}$. (3 points)
   b. Now suppose that $D$ is a primary ideal and that it is not the intersection of any proper subset of $\{A_1, A_2, \ldots, A_n\}$. Show that $\sqrt{A_i} = \sqrt{D}$ for all $i$. (7 points)

3. Let $K \subseteq E$ be a finite degree extension of fields of characteristic 0, and let $F_1$ and $F_2$ be intermediate fields. These intermediate fields are said to be linearly disjoint over $K$ if $|\langle F_1, F_2 \rangle : K| = |F_1 : K||F_2 : K|$, where $\langle F_1, F_2 \rangle$ is the subfield of $E$ generated by $F_1$ and $F_2$.
   a. Prove that $|\langle F_1, F_2 \rangle : F_i| \leq |F_2 : K|$ for any $F_1$ and $F_2$. (3 points)
   b. If $|F_1 : K|$ and $|F_2 : K|$ are relatively prime, prove that $F_1$ and $F_2$ are linearly disjoint over $K$. (2 points)
   c. Give an example with $|F_1 : K| = 2 = |F_2 : K|$ to show that fields can be linearly disjoint without having relatively prime degrees. (2 points)
   d. If $F_1$ and $F_2$ are linearly disjoint and Galois over $K$, prove that the Galois groups satisfy $\text{Gal}(\langle F_1, F_2 \rangle / K) \cong \text{Gal}(F_1 / K) \times \text{Gal}(F_2 / K)$. (3 points)

4. Let $V$ be a complex vector space, not necessarily of finite dimension. Suppose that $A, B : V \to V$ are nonzero $C$-linear transformations with $AB = \lambda BA$ for some fixed nonzero complex number $\lambda$. Assume that no proper subspace of $V$ is invariant under both $A$ and $V$. That is, if $W$ is a subspace of $V$ with $AW \subseteq W$ and $BW \subseteq W$, then $W = 0$ or $V$.
   a. Show that $A$ and $B$ are both one-to-one and onto. (5 points)
   b. If $V$ is finite dimensional, prove that $\lambda$ is a root of unity. (3 points)
   c. Show that a finite-dimensional example exists with $\lambda = -1$. (2 points)

5. Let $R$ be a ring and let $Z$ denote its center. A derivation $D : R \to R$ is a map satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in R$.
   a. If $r \in R$, show that the map $A_r : R \to R$ given by $A_r(a) = ar - ra$, for all $a \in R$, is a derivation of $R$. (3 points)
   b. If $D$ is a derivation of $R$, prove that $D(Z) \subseteq Z$. (3 points)
   c. If $D$ is a derivation of $R$ and $e \in Z$ is an idempotent, prove that $D(e) = 0$. (Hint. You may need to evaluate $(1 - 2e)^2$.) (4 points)
Algebra Qualifying Exam
August 2001

Do all 5 problems.

1. Let $G$ be a finite group of order $504 = 2^3 \cdot 3^2 \cdot 7$.
   a. Show that $G$ cannot be isomorphic to a subgroup of the alternating group $\text{Alt}_7$. (5 points)
   b. If $G$ is simple, determine the number of Sylow 3-subgroups of $G$. (5 points)

2. Let $R$ be a commutative ring with 1 and let $M$ be a maximal ideal of $R$.
   a. Show that the ring $R/M^2$ has no idempotents other than 0 and 1. (4 points)
   b. We know that $M/M^2$ is naturally an $R/M$-module. If $R$ is Noetherian, prove that this module is finitely generated. (2 points)
   c. Finally, assume that $R = K[x_1, x_2, \ldots, x_t]$ is a polynomial ring in finitely many variables over the field $K$. Prove that $\dim_K(R/M^2) < \infty$. (4 points)

3. Let $F \subseteq E$ be fields and suppose $0 \neq \alpha \in E$ with $E = F[\alpha]$. Assume that some power of $\alpha$ lies in $F$ and let $n$ be the smallest positive integer such that $\alpha^n \in F$.
   a. If $\alpha^m \in F$ with $m > 0$, show that $m$ is a multiple of $n$. (2 points)
   b. If $E$ is a separable extension of $F$, prove that the characteristic of $F$ does not divide $n$. (4 points)
   c. If every root of unity in $E$ lies in $F$, show that $|E : F| = n$. (4 points)

4. Let $A$ be a real $n \times n$ matrix. We say that $A$ is a difference of two squares if there exist real $n \times n$ matrices $B$ and $C$ with $BC = CB = 0$ and $A = B^2 - C^2$.
   a. If $A$ is a diagonal matrix, show that it is a difference of two squares. (3 points)
   b. If $A$ is a symmetric matrix that is not necessarily diagonal, again show that it is a difference of two squares. (3 points)
   c. Suppose $A$ is a difference of two squares, with corresponding matrices $B$ and $C$ as above. If $B$ has a nonzero real eigenvalue, prove that $A$ has a positive real eigenvalue. (4 points)

5. Let $K$ be a field of characteristic 0 and view the polynomial ring $V = K[x]$ as a $K$-vector space. Let $M: V \to V$ be the linear operator given by multiplication by $x$, so that $M(x^n) = x^{n+1}$ for all integers $n \geq 0$. In addition, let $D: V \to V$ be the linear operator given by differentiation with respect to $x$, so that $D(x^n) = nx^{n-1}$ for all $n \geq 0$. Let $L$ denote the set of all linear operators of the form $M^iD^j$ with $i, j \geq 0$, where $M^0 = D^0 = I$ is the identity operator on $V$.
   a. Prove that $DM - MD = I$. (3 points)
   b. Show that $L$ is a $K$-linearly independent set. (4 points)
   c. For all nonnegative integers $t$, prove that $DM^t$ is in the $K$-linear span of the set $L$. (3 points)
Algebra Qualifying Exam
August 2002

Do all 5 problems.

1. For any finite group $G$ and prime $p$, we let $n_p(G)$ denote the number of Sylow $p$-subgroups of $G$. Now suppose $K \triangleleft G$, and let $P$ be a Sylow $p$-subgroup of $G$.
   a. Show that $n_p(G/K)$ divides $n_p(G)$. (5 points)
   b. Prove that $n_p(G/K) = n_p(G)$ if and only if $P \triangleleft PK$. (5 points)

2. Let $R$ be a commutative ring with 1, and recall that a proper ideal $I \triangleleft R$ is said to be primary if, for all $r, s \in R$, the inclusion $rs \in I$ implies that either $r \in I$ or $s^n \in I$ for some integer $n \geq 1$. Assume now that every proper ideal of $R$ is primary.
   a. If $P$ is a prime ideal of $R$ and if $I \triangleleft R$, prove that either $I \subseteq P$ or $P = IP \subseteq I$. (4 points)
   b. If $M$ is a maximal ideal of $R$, prove that $M$ is precisely the set of nonunits of $R$. (3 points)
   c. Show that a proper ideal $J$ of $R$ is prime if and only if, for all $r \in R$, the inclusion $r^2 \in J$ implies that $r \in J$. (3 points)

3. Let $F$ be a field with algebraic closure $\overline{F}$, let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$, and let $E \supseteq F$ be the splitting field of $f(x)$ over $F$ with $E \subseteq \overline{F}$. Assume that $f(x)$ has $n$ distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $E$.
   a. Show that there exists an element $\beta \in E$ and $n$ polynomials $p_i(x) \in F[x]$ with $p_i(\beta) = \alpha_i$ for all $i = 1, 2, \ldots, n$. (3 points)
   b. Continuing with the notation of (a), let $g(x) \in F[x]$ be the minimal polynomial of $\beta$ over $F$. If $\gamma \in \overline{F}$ is any root of $g(x)$, show that $p_1(\gamma), p_2(\gamma), \ldots, p_n(\gamma)$ are equal to $\alpha_1, \alpha_2, \ldots, \alpha_n$ in some order. (4 points)
   c. Continuing with the notation of (a) and (b), if $\gamma$ and $\gamma'$ are both roots of $g(x)$ and if $p_i(\gamma) = p_i(\gamma')$ for all $i = 1, 2, \ldots, n$, show that $\gamma = \gamma'$. (3 points)

4. Let $A$ be an $n \times n$ matrix over an algebraically closed field $K$.
   a. Show that $A = B + C$ where $B$ is a diagonalizable matrix, $C$ is nilpotent with $C^n = 0$, and $BC = CB$. (4 points)
   b. If char $K = p > 0$, prove that $A^{p^t}$ is diagonalizable for some integer $t \geq 0$. (2 points)
   c. If $K$ is the complex number field, prove that the exponential matrix $\exp(A) = \sum_{k=0}^{\infty} A^k/k!$ exists. (4 points)

(over)
5. Let $R$ be a ring with 1, let $V$ be a right $R$-module, and let $W$ be a submodule of $V$. Suppose that $V = V_1 + V_2 + \cdots + V_n = \sum_i V_i$ is an internal direct sum of the simple (that is, irreducible) submodules $V_1, V_2, \ldots, V_n$. Furthermore, for each subscript $j$, let $V_j' = \sum_{i \neq j} V_i$ be the internal direct sum of those $V_i$ with $i \neq j$, so that $V = V_j + V_j'$.

a. If $W \neq \emptyset$, prove that $W$ contains a minimal proper submodule and a maximal proper submodule. (3 points)

b. If $W$ is a maximal proper submodule of $V$, prove that there exists a subscript $k$ with $V = W + V_k$ and hence that $V/W \cong V_k$. (3 points)

c. If $W$ is simple, show that there exists a subscript $j$ with $W \cong V_j$ and $W + V_j' = V$. (4 points)
Algebra Qualifying Exam
August 2003

Do all 5 problems.

1. Let $G$ be a finite group of order $504 = 2^3 \cdot 3^2 \cdot 7$.
   a. Show that $G$ cannot be isomorphic to a subgroup of the alternating group $Alt_7$. (5 points)
   b. If $G$ is simple, determine the number of Sylow 3-subgroups of $G$. (5 points)

2. Let $R$ be a commutative integral domain with 1.
   a. Let $K$ be the field of fractions of $R$, let $t \in R$ be a nonzero element, and suppose that $K = R[1/t]$. In other words, every element of $K$ can be written as a polynomial in $1/t$ with coefficients in $R$. Show that $t$ is contained in every nonzero prime ideal of $R$. (5 points)
   b. Now suppose that $R$ is the polynomial ring $R = F[X_1, X_2, \ldots, X_n]$ where $F$ is an infinite field. If $f(X_1, X_2, \ldots, X_n)$ is contained in every nonzero prime ideal of $R$, show first that $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n \in F$. Then prove that the latter zero-value property implies that $f$ is the zero polynomial. (5 points)

3. Let $F \subseteq E$ be fields and suppose $0 \neq \alpha \in E$ with $E = F[\alpha]$. Assume that some power of $\alpha$ lies in $F$ and let $n$ be the smallest positive integer such that $\alpha^n \in F$.
   a. If $\alpha^m \in F$ with $m > 0$, show that $m$ is a multiple of $n$. (2 points)
   b. If $E$ is a separable extension of $F$, prove that the characteristic of $F$ does not divide $n$. (4 points)
   c. If every root of unity in $E$ lies in $F$, show that $|E : F| = n$. (4 points)

4. Let $A$ be a real $n \times n$ matrix. We say that $A$ is a *difference of two squares* if there exist real $n \times n$ matrices $B$ and $C$ with $BC = CB = 0$ and $A = B^2 - C^2$.
   a. If $A$ is a diagonal matrix, show that it is a difference of two squares. (3 points)
   b. If $A$ is a symmetric matrix that is not necessarily diagonal, again show that it is a difference of two squares. (3 points)
   c. Suppose $A$ is a difference of two squares, with corresponding matrices $B$ and $C$ as above. If $B$ has a nonzero real eigenvalue, prove that $A$ has a positive real eigenvalue. (4 points)

5. Let $K$ be a field of characteristic 0 and view the polynomial ring $V = K[x]$ as a $K$-vector space. Let $M: V \to V$ be the linear operator given by multiplication by $x$, so that $M(x^n) = x^{n+1}$ for all integers $n \geq 0$. In addition, let $D: V \to V$ be the linear operator given by differentiation with respect to $x$, so that $D(x^n) = nx^{n-1}$ for all $n \geq 0$. Let $L$ denote the set of all linear operators of the form $M^iD^j$ with $i, j \geq 0$, where $M^0 = D^0 = I$ is the identity operator on $V$.
   a. Prove that $DM - MD = I$. (3 points)
   b. Show that $L$ is a $K$-linearly independent set. (4 points)
   c. For all nonnegative integers $t$, prove that $DM^t$ is in the $K$-linear span of the set $L$. (3 points)
Algebra Qualifying Exam
August 2004

Do all 5 problems.

1. Let $G$ be a finite group of order $pm$, where $p$ is a prime that does not divide $m$, and let $n$ denote the number of Sylow $p$-subgroups of $G$.
   a. Show that the exists a homomorphism $\theta$ from $G$ to the symmetric group $\text{Sym}(n)$ such that, for all $x \in G$ of order $p$, the image $\theta(x)$ has exactly one fixed point. (4 points)
   b. Now suppose that $G$ is simple and contains an element $y$ of order $pq$, for some prime $q \neq p$. If $\theta$ is as in part (a), show that $\theta(y)$ must contain a cycle of length $pq$ in its cycle decomposition. (3 points)
   c. Now let $p = 5$ and suppose that $G$ is a simple group of order 660. Show that $G$ has no element of order 15. (3 points)

2. Let $R$ be a ring with 1, let $M$ be a finitely generated right $R$-module, and let $N < M$ be a proper submodule of $M$.
   a. Prove that there exists a maximal submodule $K$ of $M$ with $N \subseteq K < M$. (5 points)
   b. Show that $N + MJ < M$, where $J = J(R)$ denotes the Jacobson radical of $R$. (5 points)

3. In the field $\mathbb{C}$ of complex numbers, let $Q$ be the subfield of rational numbers, let $i = \sqrt{-1}$, and let $\sqrt{2}$ be the positive real fourth root of 2.
   a. Prove that the polynomial $X^4 - 2$ is irreducible over the field $Q[i]$. (4 points)
   b. If $\sqrt{2} + i$ is a root of a polynomial $f(X) \in Q[X]$, show that $i\sqrt{2} + i$ is also a root of that polynomial. (3 points)
   c. Compute the degree of the minimal polynomial of $\sqrt{2} + i$ over $Q$. (3 points)

4. Let $V$ be a vector space of dimension $n$ over a field $K$. Suppose $V$ is spanned by the $n + 1$ vectors $v_0, v_1, \ldots, v_n$ where $v_0 + v_1 + \cdots + v_n = 0$. Now let $W$ be a second $K$-vector space and let $w_0, w_1, \ldots, w_n \in W$. Find necessary and sufficient conditions on the elements $w_0, w_1, \ldots, w_n$ so that there exists a linear transformation $T: V \to W$ with $T(v_i) = w_i$ for $i = 0, 1, \ldots, n$. (10 points)

5. Let $k$ be a field, let $K = k(x, y)$ be the rational function field over $k$ in the indeterminates $x$ and $y$, and let $\overline{K}$ denote the algebraic closure of $K$. Suppose $s$ and $t$ are elements of $\overline{K}$ with $s^2 = x + y$ and $t^3 = xy$, and let $R = k[s, t]$ be the subring of $\overline{K}$ generated by $k$, $s$ and $t$. Show that every element $r \in R$ is the root of some irreducible monic polynomial $f(Z) \in K[Z]$ of degree at most 6 with all coefficients in the polynomial ring $k[x, y]$. (10 points)
1. Let \( G \) be a finite group and let \( H, K \) and \( L \) be normal subgroups of \( G \) with \( H \cap K = H \cap L = K \cap L = 1 \).

(a) If \(|HKL| < |H| \cdot |K| \cdot |L|\), show that \( H, K \) and \( L \) each have a central subgroup of prime order \( p \), for the same \( p \). Of course, the central subgroups must be different. (6 points)

(b) Show by example that each prime \( p \) can occur in the above context. In other words, find a group \( G \) having pairwise disjoint normal subgroups \( H, K \) and \( L \), such that \(|HKL| < |H| \cdot |K| \cdot |L|\) and \( p \) divides the orders of \( Z(H), Z(K) \) and \( Z(L) \). (4 points)

2. Let \( R \) be a ring with 1 and let \( V \) be a right \( R \)-module. Assume that \( V \) is Noetherian, so that every ascending chain of submodules must terminate. Now let \( \theta : V \rightarrow V \) be an \( R \)-homomorphism.

(a) Show that \( \ker(\theta^{n+1}) = \ker(\theta^n) \) for some integer \( n \geq 1 \). (3 points)

(b) If \( \theta \) is onto, prove that it is one-to-one. (3 points)

(c) Suppose that \( V \) has a unique maximal submodule \( M \). Furthermore assume that if \( X \subseteq Y \) are any submodules of \( V \) with \( Y/X \cong V/M \), then we must have \( Y = V \). Prove that \( \theta \) is either 0 or an isomorphism. (4 points)

3. Let \( p \) and \( q \) be different prime numbers and consider the positive real numbers \( \mu = \sqrt[q]{p} = p^{1/q} \) and \( \nu = \sqrt[p]{q} = q^{1/p} \).

(a) Let \( F \) be a subfield of the real numbers \( \mathbb{R} \) that does not contain \( \mu \). If \( \mu^n \in F \) for some positive integer \( n \), show that \( p \) divides \( n \). (3 points)

(b) Again, let \( F \subseteq \mathbb{R} \) be a field not containing \( \mu \). Show that \(|F[\mu] : F| = p \). (Hint. Consider the constant term of the minimal polynomial of \( \mu \) over \( F \).) (4 points)

(c) Now let \( F = \mathbb{Q}[\mu + \nu] \) be the field extension of the rationals \( \mathbb{Q} \) generated by \( \mu + \nu \). Show that \(|F : \mathbb{Q}| = pq \). (3 points)

4. Let \( F \) be a field and let \( A \) and \( B \) be nonsingular \( 3 \times 3 \) matrices over \( F \). Suppose that \( B^{-1}AB = 2A \).

(a) Find the characteristic of \( F \). (3 points)

(b) If \( n \) is a positive or negative integer not divisible by 3, prove that the matrix \( A^n \) has trace 0. (3 points)

(c) Prove that the characteristic polynomial of \( A \) is \( X^3 - a \) for some \( a \in F \). (4 points)

5. Let \( G = GL(n, \mathbb{Z}) \) be the group of invertible \( n \times n \) matrices with entries in the integers \( \mathbb{Z} \). Fix a prime number \( p \) and let \( S \) be the subset of \( G \) consisting of matrices of the form \( I + pX \), where \( I \) is the identity matrix and where \( X \) is some (not necessarily invertible) integer matrix. Note that, if \( M \) is a nonidentity element of \( S \), then \( M = I + tY \) where \( t \geq 1 \) and where some entry of the integer matrix \( Y \) is not divisible by \( p \).

(a) Prove that \( S \) is a subgroup of \( G \). (3 points)

(b) Suppose \( M \in S \) has prime order \( q \). Show that \( q = p \). (3 points)

(c) If \( p > 2 \), show that \( S \) has no element of order \( p \) and deduce that \( S \) has no nonidentity element of finite order. (4 points)
1. Let $M$ be a minimal normal subgroup of the finite group $G$ and let $N/M$ be a nontrivial normal subgroup of $G/M$. Assume that $M$ is a $p$-group and that $N/M$ is a $q$-group for some primes $p$ and $q$, not necessarily distinct.
   a. Show that $G = MH$ where $H$ is a subgroup of $G$ having a nontrivial normal $q$-subgroup. (4 points)
   b. If $M$ is self-centralizing in $G$, prove that $p \# q$. (3 points)
   c. If $M$ is self-centralizing and if $H$ is as in part a, prove that $M \cap H = 1$. (3 points)

2. Let $R$ be a ring with 1, not necessarily commutative. Recall that an element $e$ of $R$ is an idempotent if $e^2 = e$, and an element $0 \neq r \in R$ is a zero divisor if there exists $0 \neq s \in R$ with $rs = 0$ or $sr = 0$. Now assume that $R$ has a nil ideal $N$ such that $R/N$ has no zero divisors.
   a. Show that the only idempotents of $R$ are the elements 0 and 1. (5 points)
   b. If $R/N$ is a division ring, prove that every zero divisor in $R$ is nilpotent. (5 points)

3. Let $C \supseteq E \supseteq K \supseteq Q$ be a chain of fields, where $C$ is the field of complex numbers, $Q$ is the field of rational numbers, $E = Q[\alpha]$ with $\alpha^n \in Q$, and $K$ is generated by all roots of unity in $E$. Assume that $E$ is a Galois extension of $Q$.
   a. Show that the Galois group $\text{Gal}(E/K)$ is cyclic. (5 points)
   b. If the restriction $\tau$ of complex conjugation to $E$ is in the center of $\text{Gal}(E/Q)$, prove that $|\alpha|^2 \in Q$, where $| \cdot |$ denotes complex absolute value. (5 points)

4. Let $V \neq 0$ be a finite dimensional vector space over a field $F$ and let $T: V \to V$ be a linear transformation. We say that $T$ is regular if its characteristic polynomial and minimal polynomial are equal.
   a. If there exists a vector $v \in V$ such that $V$ is spanned by $v, T(v), T^2(v), \ldots$, prove that $T$ is regular. (5 points)
   b. Assume that $T$ is regular and let $W$ be a subspace of $V$ with $T(W) \subseteq W$. Show that $T_W$, the restriction of $T$ to $W$, and $T_{V/W}$, the induced action of $T$ on $V/W$, are both regular. (5 points)

5. Let $F = \text{GF}(q)$ be the finite field with $q$ elements and let $M_2(F)$ be the ring of $2 \times 2$ matrices over $F$.
   a. If $A \in M_2(F)$ has equal eigenvalues in the algebraic closure of $F$, show that the eigenvalues of $A$ actually belong to $F$. (4 points)
   b. Determine the number of nonzero nilpotent matrices in $M_2(F)$ as a function of $q$. (Hint. Use Jordan canonical form and note that the group $G$ of invertible $2 \times 2$ matrices over $F$ has order $(q^2 - 1)(q^2 - q)$.) (6 points)
Algebra Qualifying Exam  
August 2007

Do all 5 problems.

1. Let $G$ be a finite group of order $|G| = 504 = 2^3 \cdot 3^2 \cdot 7$.
   a. If $G$ has a normal subgroup $N$ of order 8, show that $G$ has at most 8 Sylow 7-subgroups, that is $|\text{Syl}_7(G)| \leq 8$. (5 points)
   b. If $|\text{Syl}_7(G)| \leq 8$, prove that $G$ has an element of order 21. (4 points)
   c. If $G$ is isomorphic to a subgroup of $\text{Sym}_9$, the symmetric group of degree 9, show that $G$ cannot have a normal subgroup of order 8. (1 point)

2. Let $R$ be a commutative integral domain with field of fractions $F$, and assume that $R$ is integrally closed.
   a. Suppose $K$ is a field containing $F$ and let $\alpha \in K$ be integral over $R$. Show that the minimal monic polynomial of $\alpha$ over $F$ is contained in $R[\alpha]$. (5 points)
   b. Let $f(x) \in R[x]$ be a monic polynomial. Show that $f(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. (5 points)

3. Let $F$ be a field of characteristic 0 and let $E$ be a finite Galois extension of $F$.
   a. If $0 \neq \alpha \in E$ with $E = F[\alpha]$, show that $F[\alpha^2] \neq E$ if and only if there exists an automorphism $\sigma \in \text{Gal}(E/F)$ with $\alpha^\sigma = -\alpha$. (6 points)
   b. Prove that there exists an element $\alpha \in E$ with $E = F[\alpha^2]$. (4 points)

4. Let $V$ be a finite-dimensional vector space over the field $F$ with $\dim_F V = n$, and let $(\ , \ ): V \times V \to F$ be a symmetric bilinear form. If $X$ is a subset of $V$, write $X^\perp = \{v \in V \mid (X, v) = 0\}$ for the subspace of $V$ perpendicular to $X$.
   a. If $W$ is a subspace of $V$, show that $\dim_F W + \dim_F W^\perp \geq \dim_F V$. (Hint. If $w \in W$, note that $\{w\}^\perp$ has codimension $\leq 1$ in $V$.) (2 points)
   b. Now suppose $(\ , \ )$ is nonsingular, so that $V^\perp = 0$. If $A = \{a_1, a_2, \ldots, a_n\}$ is a basis for $V$, prove that there exists a unique dual basis $A' = \{a'_1, a'_2, \ldots, a'_n\}$. That is, $A'$ is a basis with $(a_i, a'_j) = 0$ if $i \neq j$ and $(a_i, a'_i) = 1$. (4 points)
   c. Again suppose $(\ , \ )$ is nonsingular, and let $B = \{b_1, b_2, \ldots, b_n\}$ be a second basis for $V$ with dual basis $B' = \{b'_1, b'_2, \ldots, b'_n\}$. Compare the change of basis matrix from $A$ to $B$ with the change of basis matrix from $B'$ to $A'$. (4 points)

5. Let $R$ be a not necessarily commutative ring with 1.
   a. If $V_1, V_2, \ldots, V_n$ are $n$ nonisomorphic irreducible right $R$-modules, show that there exists an $R$-module epimorphism from $R$, viewed as a right $R$-module, to the external direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_n$. (5 points)
   b. Suppose $R$, viewed as a right $R$-module, has a finite composition series with nonisomorphic composition factors. Prove that the Jacobson radical of $R$ is equal to 0. (5 points)
1. In this problem we prove that a Sylow 2-subgroup of a simple group of order 168 is its own normalizer.
   a. If $G$ is a group of order 24 and $G$ has a normal Sylow 2-subgroup, show that $G$ contains an element of order 6. (4 points)
   b. If $G$ is a simple group and $H$ is a subgroup of $G$ with $|G : H| = 7$, show that $H$ contains no element of order 6. (3 points)
   c. Let $G$ be a simple group with $|G| = 168$ and let $P$ be a Sylow 2-subgroup of $G$. Prove that $N_G(P) = P$. (3 points)

2. Let $\mathbb{Z}$ be the ring of integers and let $S = \mathbb{Z} \oplus \mathbb{Z}$ be the ring external direct sum of two copies of $\mathbb{Z}$. Now let $R$ be the subring of $S$ given by
   \[ R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \mod 6 \}. \]
   a. Show that $R$ is a finitely generated $\mathbb{Z}$-module and conclude that $R$ is a Noetherian ring. (3 points)
   b. Prove that the ideal $P$ of $R$ given by
   \[ P = \{(a, 0) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv 0 \mod 6 \} \]
   is prime. (2 points)
   c. If $Q$ is a primary ideal of $R$ with $P = \sqrt{Q}$, the radical of $Q$, show that $Q = P$. (5 points)

3. Let $\mathbb{C}$ denote the complex number field and let $E \subseteq \mathbb{C}$ be the splitting field over the rational numbers $\mathbb{Q}$ of the polynomial $x^3 - 2$.
   a. Show that $|E : \mathbb{Q}| = 6$. (2 points)
   b. If $\alpha \in E$ and $\alpha^5 \in \mathbb{Q}$, prove that $\alpha \in \mathbb{Q}$. (5 points)
   c. Show that there exists $\beta \in E$ with $\beta^2 \in \mathbb{Q}$, but $\beta \notin \mathbb{Q}$. (3 points)
4. Let $S$, $T$ and $M$ be $n \times n$ matrices over the complex numbers $\mathbb{C}$ and suppose that $SM = MT$.

a. If $f(x) \in \mathbb{C}[x]$ is the minimal polynomial of $T$, show that $f(S)M = 0$. (4 points)

b. If $M \neq 0$, deduce that $S$ and $T$ have a common eigenvalue. (3 points)

c. Now suppose $n = 2$,

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}.$$  

Find a nonzero matrix $M$ with $SM = MT$ and show that it is impossible to find an invertible matrix $M$ with this property. (3 points)

5. Let $R$ be a subring of the ring $\mathbb{M}_n(\mathbb{C})$ of all complex $n \times n$ matrices, and suppose that $R$ is finitely generated as module over the integers $\mathbb{Z}$. Let $M \in R$.

a. Show that $M$ is contained in a commutative subring $S$ of $\mathbb{M}_n(\mathbb{C})$ that is finitely generated as a $\mathbb{Z}$-module. (3 points)

b. Deduce that there is a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(M) = 0$. (2 points)

c. Prove that $\text{tr}(M)$, the matrix trace of $M$, is an algebraic integer. (5 points)
1. Let $H$ be a maximal subgroup of the finite group $G$ and let $\mathcal{X}$ be the set of normal subgroups $X$ of $G$ such that $X \neq 1$ and $X \cap H = 1$.
   a. Show that all members of $\mathcal{X}$ are minimal normal subgroups of $G$ of the same order. (3 points)
   b. If some member of $\mathcal{X}$ is abelian, show that all members of $\mathcal{X}$ are abelian $p$-groups for some prime $p$. (3 points)
   c. Let $U, V \in \mathcal{X}$ be distinct and assume that $\mathcal{X}$ contains at least one additional member different from $U$ and $V$. Show that $(UV \cap H) \triangleleft G$ and conclude that $(UV \cap H) \subseteq \mathbb{Z}(UV)$. (4 points)

2. Let $R \subseteq S$ be commutative rings with the same 1, and assume that every element of $S$ is integral over $R$.
   a. If $r \in R$ has an inverse in $S$, prove that this inverse is contained in $R$. (3 points)
   b. Suppose $R$ is a field and let $s \in S$ be a regular element (that is, if $sx = 0$ for some $x \in S$, then $x = 0$). Show that $s$ is invertible in $S$. (3 points)
   c. If $P$ is a prime ideal of $S$, prove that $P$ is a maximal ideal of $S$ if and only if $R \cap P$ is a maximal ideal of $R$. (4 points)

3. Let $F$ be a field and let $f(x) \in F[x]$ be an irreducible polynomial with splitting field $E$ over $F$. Choose $\alpha \in E$ with $f(\alpha) = 0$. Furthermore, for some fixed integer $n \geq 1$, let $g(x)$ be an irreducible polynomial in $F[x]$ with $g(\alpha^n) = 0$.
   a. Show that $\deg(g)$ divides $\deg(f)$ and that $\deg(f)/\deg(g) \leq n$. (5 points)
   b. If $\deg(f)/\deg(g) = n$ and if the characteristic of $F$ does not divide $n$, prove that $E$ contains a primitive $n$th root of unity. (5 points)

4. Let $V$ be a vector space over a field $F$ and let $(,): V \times V \rightarrow F$ be a bilinear form. For each $x \in V$ define $A(x) = \{ y \in V \mid (x, y) = -(y, x) \}$. Now suppose $v$ is a fixed element of $V$ with $(v, v) \neq 0$.
   a. For all $x \in V$, show that $A(x)$ is a subspace of $V$ of codimension at most 1. (4 points)
   b. If the characteristic of $F$ is different from 2, prove that $A(v)$ is a subspace of $V$ of codimension precisely 1. (1 point)
   c. If $F$ is algebraically closed and has characteristic different from 2, show that either $(a, a) = 0$ for every element $a \in A(v)$, or there exists $y \in V \setminus A(v)$ with $(y, y) = 0$. (5 points)

5. A multiplicative abelian group $A$ is said to be “divisible” if, for all $a \in A$ and positive integers $n$, there exists $b \in A$ with $b^n = a$.
   a. If $A$ is divisible and $\overline{A}$ is a homomorphic image of $A$, prove that $\overline{A}$ is divisible. (2 points)
   b. If $A$ is a finite divisible group, prove that $A = 1$. (3 points)
   c. Suppose $A$ is divisible and that $A$ is a subgroup of the abelian group $B$. If $A \cap X > 1$ for all nonidentity subgroups $X$ of $B$, prove that $A = B$. (5 points)
1. Let $G$ be a finite group and let $N$ be a minimal normal subgroup of $G$. Suppose $N = S_1 \times S_2 \times \cdots \times S_r$, where each $S_i$ is a simple subgroup and where $S_1$ is not abelian.

(a) Show that $Z(N) = 1$, where $Z(N)$ is the center of $N$, and deduce that each $S_i$ is nonabelian. (3 points)

(b) If $g \in G$, show that $(S_1)^g = S_i$ for some $i = 1, 2, \ldots, r$. (4 points)

(c) Prove that $G$ has a subgroup of index $r$. (3 points)

2. Let $R$ be a commutative ring with 1, and let $P$ be a prime ideal of $R$. Write $M$ to denote the set of elements of $R$ that are not in $P$, and recall that $M$ is a multiplicatively closed subset of $R$. For each ideal $I$ of $R$, with $I$ possibly equal to $R$, define $I' = \{ r \in R \mid rm \in I \text{ for some } m \in M \}$.

Thus $I'$ is an ideal of $R$ with $I' \supseteq I$.

(a) Prove that $(I')' = I'$. (In other words, the map $I \mapsto I'$ is a closure operator.) Furthermore, show that $I' = R$ if and only if $I \cap M \neq \emptyset$. (3 points)

(b) If $Q$ is a primary ideal of $R$, show that either $Q' = Q$ or $Q' = R$. In particular, deduce that $P' = P$. (3 points)

(c) If $P \supseteq I \supseteq P^n$ for some integer $n \geq 1$, prove that $I'$ is a primary ideal with radical equal to $P$. (4 points)

3. Let $\mathbb{Q}$ be the field of rational numbers and let $f \in \mathbb{Q}[X]$. We say that $f$ is a “special” polynomial provided that $f$ is irreducible in $\mathbb{Q}[X]$, its degree is at least 2, and $f$ splits over $\mathbb{Q}[\alpha]$, where $\alpha$ is some root of $f$ in some extension field of $\mathbb{Q}$.

(a) Let $f \in \mathbb{Q}[X]$ be an irreducible polynomial with degree at least 2, and let $L$ be a splitting field for $f$ over $\mathbb{Q}$. If Gal($L/\mathbb{Q}$) is abelian, prove that $f$ is special. (4 points)

(b) Suppose $L$ is a finite degree Galois extension of $\mathbb{Q}$ with $L$ strictly larger than $\mathbb{Q}$. Show that there exists a special polynomial $f$ having a root in $L$. (3 points)

(c) Prove that the polynomial $X^n - 2 \in \mathbb{Q}[X]$ is not special if $n \geq 3$. (3 points)

(more over)
4. Let $V$ be an $n$-dimensional vector space over the field $\mathbb{R}$ of real numbers, and let $\theta : V \times V \to \mathbb{R}$ be a bilinear form. Given an (ordered) basis $\mathcal{A} = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$ of $V$ define the $n \times n$ real matrix $M_\mathcal{A}$ to be $M_\mathcal{A} = [\theta(\alpha_i, \alpha_j)]$.

(a) Show that there exists a nonzero vector $v \in V$ with $\theta(v, V) = 0$ if and only if $M_\mathcal{A}$ is a singular matrix. (3 points)

(b) If $\mathcal{B} = \{ \beta_1, \beta_2, \ldots, \beta_n \}$ is a second (ordered) basis, we can write $\beta_i = \sum_j p_{i,j} \alpha_j$ for suitable $p_{i,j} \in \mathbb{R}$, where the matrix $P = [p_{i,j}]$ is the change-of-basis matrix. Describe $M_\mathcal{B}$ in terms of $M_\mathcal{A}$ and the change of basis matrix $P$. (3 points)

(c) Now assume that $\theta$ is symmetric and positive definite. Prove that $M_\mathcal{A} = QQ^T$ for some nonsingular matrix $Q$. Here $Q^T$ is the transpose of $Q$. (4 points)

5. Let $R$ be a ring (with 1) that is not necessarily commutative, and let $M$ be a right $R$-module. Suppose that $M$ has a submodule $N$ that is maximal with the property of being noetherian.

(a) Show that no nonzero submodule of $M/N$ is either artinian or noetherian (3 points)

(b) If $R$ is either a right artinian or right noetherian ring, prove that $M$ is noetherian. In other words, show that $M = N$. (3 points)

(c) If $M = R$ viewed as a right $R$-module, deduce that $N$ is a 2-sided ideal of $R$. (4 points)
Algebra Qualifying Exam
August 2011

Do all 5 problems.

1. Let \( p \) be a fixed prime number and let \( G \) be a finite group. A normal subgroup \( K \) of \( G \) is said to be a “normal \( p \)-complement” if \( p \nmid |K| \) and \( |G : K| \) is a power of \( p \).
   (a) If \( G \) has a normal \( p \)-complement and \( H \) is a subgroup of \( G \), show that \( H \) has a normal \( p \)-complement. (3 points)
   (b) If \( G \) has a normal \( p \)-complement and \( N \) is a normal subgroup of \( G \), show that \( G/N \) has a normal \( p \)-complement. (3 points)
   (c) Let \( U \) and \( V \) be normal subgroups of \( G \), and suppose both \( U \) and \( V \) have normal \( p \)-complements. Prove that \( UV \) has a normal \( p \)-complement. (4 points)

2. Let \( R \) be a commutative ring with 1 and let \( Q \) be a primary ideal of \( R \). Suppose \( a \in R \) with \( a \notin Q \), and define \( I = \{ r \in R \mid ar \in Q \} \). (Thus \( I \) is an ideal of \( R \).)
   (a) Show that \( \sqrt{I} = \sqrt{Q} \). (3 points)
   (b) Prove that \( I \) is a primary ideal of \( R \). (3 points)
   (c) If \( R \) is noetherian, show that the element \( a \) can be chosen so that \( I \) is a prime ideal. (4 points)

3. Let \( K \subseteq F \subseteq E \) be fields with \( E = F[\alpha] \) and with \( \alpha^n \in F \) for some positive integer \( n \). Suppose \( K \) contains a primitive \( n \)th root of unity, and let \( L \) be a field with \( K \subseteq L \subseteq E \) and \( L \cap F = K \).
   (a) If \( L \) is Galois over \( K \), show that \( L = K[\beta] \) for some element \( \beta \) with \( \beta^n \in K \). (6 points)
   (b) Show by example that \( L \) need not be Galois over \( K \). (Hint. Take \( n = 2 \).) (4 points)

4. Let \( V \) be an \( n \)-dimensional vector space over a field \( K \). Let \( m \) be a positive integer, and write \( V^m = V \times V \times \cdots \times V \) (\( m \) times). A “multilinear, alternating” function \( f: V^m \to K \) is a function \( f(x_1, x_2, \ldots, x_m) \) which is multilinear, which means that it is linear in each of the \( m \) variables \( x_i \in V \), and alternating, which means that \( f(x_1, x_2, \ldots, x_m) = 0 \) if any two of the variables are equal. Let \( S \) be the set of all multilinear, alternating functions from \( V^m \) to \( K \).
   (a) Prove that \( S \) is naturally a \( K \)-vector space and that \( S = 0 \) if \( m > n \). (3 points)
   (b) If \( m \leq n \), show that \( \dim_K S \) is no greater than the binomial coefficient \( \binom{n}{m} \). (4 points)
   (c) Show that \( \dim_K S = 1 \) when \( m = n \). (3 points)

(more over)
5. Let $A$ be an additive abelian group. We say that $A$ is “free abelian” if it has a (possibly infinite) subset $B$ such that every element $a \in A$ is uniquely writable as a finite $\mathbb{Z}$-linear combination of elements of $B$. (In other words, $a$ is uniquely of the form $a = \sum_{b \in B} n_b b$, where the coefficients $n_b$ lie in $\mathbb{Z}$, and all but finitely many of them are 0.) Next, we say that $a \in A$ is a “divisible element” if for each integer $n > 0$, there exists an element $a_n \in A$ such that $na_n = a$.

(a) If $A$ is free abelian, prove that $A$ contains no nonzero divisible element. (3 points)

(b) Now let $A$ be the group of infinite-tuples of integers $(x_1, x_2, x_3, \ldots)$, where addition is defined coordinatewise. Let $B$ be the subgroup of $A$ consisting of those tuples that have at most finitely many nonzero entries. Prove that $A/B$ contains a nonzero divisible element, and conclude that $A/B$ is not free abelian. (7 points)
1. Let $p$ be a prime number. For any finite group $G$, let $\mathcal{B}(G)$ denote the subgroup of $G$ generated by all the Sylow $p$-subgroups of $G$.

(a) Show that $\mathcal{B}(G)$ is the unique normal subgroup of $G$ minimal with the property that its index is not divisible by $p$. (3 points)

(b) Let $L$ be a normal subgroup of the finite group $G$. Show that $\mathcal{B}(L)$ is normal in $G$ and that $\mathcal{B}(L) = \mathcal{B}(G)$ if $|G:L|$ is not divisible by $p$. (3 points)

(c) Now let $H$ be a subgroup of the finite group $G$ with $|G:H| = p$. If $L$ is the largest normal subgroup of $G$ contained in $H$, prove that $|H:L|$ is not divisible by $p$ and deduce that $\mathcal{B}(H)$ is normal in $G$. (4 points)

2. Let $F$ be a field and let $R = F[x,y]$ be the polynomial ring over $F$ in the two indeterminates $x$ and $y$. Write $I = (x)$ for the principal ideal of $R$ generated by $x$.

(a) Prove that the $F$-vector space $I/I^2$ is infinite dimensional. (3 points)

(b) Let $S$ be the subring of $R$ given by $S = F + I$, so that $I$ is also an ideal of $S$. Prove that $I$ is not finitely generated as an ideal of $S$. (3 points)

(c) Let $M$ be a maximal ideal of $R$ and let $\theta: R \to R/M$ be the natural epimorphism. Then $\theta(S)$ is a ring with $\theta(F) \subseteq \theta(S) \subseteq \theta(R)$. Discuss the nature of the extension $\theta(F) \subseteq \theta(R)$, prove that $\theta(S)$ is a field and conclude that $M \cap S$ is a maximal ideal of $S$. (4 points)

3. Let $Q$ denote the field of rational numbers and $C$ the field of complex numbers.

(a) Suppose $K$ and $L$ are subfields of $C$, each of which is Galois over $Q$. Show that the field $E$ generated by $K$ and $L$ is Galois over $Q$. (2 points)

(b) If in part (a), the degrees $|K:Q|$ and $|L:Q|$ are coprime, show that $\text{Gal}(E/Q)$ is isomorphic to the direct product of the groups $\text{Gal}(K/Q)$ and $\text{Gal}(L/Q)$. Deduce that $|E:Q| = |K:Q||L:Q|$. (5 points)

(c) Prove that there exists a subfield $F$ of $C$ such that $F$ is Galois over $Q$ with $|F:Q| = 55$. (3 points)

4. Let $n$ be a positive integer, let $V$ be an $n$-dimensional vector space over the field $K$, and let $T: V \to V$ be a linear transformation.

(a) Suppose that there exists an element $v \in V$ such that $V$ is spanned by $v, vT, vT^2, \ldots$. Prove that the minimal polynomial of $T$ is equal to the characteristic polynomial of $T$. (4 points)

(b) As a partial converse, suppose that the characteristic polynomial of $T$ has $n$ distinct roots in $K$. Prove that there exists an element $v \in V$ such that $V$ is spanned by $v, vT, vT^2, \ldots$. (6 points)

(more over)
5. Let $R$ be a ring with 1, but not necessarily commutative. Suppose that $x^5 = x$ for all elements $x \in R$.
(a) Show that the Jacobson radical of $R$ is equal to 0. (2 points)
(b) Now assume that $R$ is right Artinian. Prove that $R$ is a finite direct sum of division rings. (3 points)
(c) Let $D$ be a division ring direct summand of $R$. If $F$ is any subfield of $D$, prove that $F$ is isomorphic to GF(2), GF(3) or GF(5). (3 points)
(d) Deduce that $D$ above is isomorphic to GF(2), GF(3) or GF(5), and conclude that $R$ is commutative. (2 points)