Problem 1.

Let \( f \) be a real valued function on the real line.

Suppose there is a constant \( \alpha \) between zero and one and a positive constant \( C \) such that for all \( y \),

\[
|f(x) - f(y)| \leq C |x - y|^{\alpha}.
\]

Prove that for all \( h > 0 \) there is a continuously differentiable function \( g_h \) such that for all \( x \),

1. \( |f(x) - g_h(x)| \leq Kh^{\alpha} \),
2. \( |g_h'(x)| \leq Kh^{\alpha-1} \),

\( K \) is a constant not depending on \( h \).  \textbf{Hint:}  Try convolution.

Problem 2.

Let \( f \) be a continuously differentiable function on \( [0, \infty) \) such that

\[
\int_0^\infty |f'(t)|^2 \, dt < \infty.
\]

Suppose that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \, dt = L.
\]

Prove that

\[
\lim_{T \to \infty} f(t) = L.
\]

Problem 3.

Prove: If \( f \in L^1(\mathbb{R}) \) then

\[
\int e^{i\lambda t^2} f(t) \, dt \to 0 \quad \text{as} \quad \lambda \to \infty.
\]
Problem 4.

Evaluate: \[ \sum_{1}^{\infty} \frac{1}{1 + n^2}. \]

Problem 5.

For \( \Re(s) > 1 \), define

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \]

Show that \((s - 1) \zeta(s)\) has an analytic continuation to \( \Re(s) > 0 \).

Problem 6.

Show:

\[ \left| \int_{-N}^{N} e^{is^2} h \, ds \right| \leq \frac{C}{\sqrt{h}}. \]

Problem 7.

Give an example of a function \( f(z) \) which satisfies:

1) \( f(z) \) is holomorphic for \( |z| < 1 \) and continuous for \( |z| \leq 1 \).
2) \( f(e^{i\theta}) \) is \( C^\infty \) in \( \theta \) and
3) \( f(z) \) is not analytic in any disk centered at the origin with radius bigger than one.
Qualifying Exam

Analysis Exam

January 1979

Do any five problems.

Problem 1. Let \( f \) be twice continuously differentiable on \( (-\infty, \infty) \).

Suppose that \( |f(x)| \leq \frac{1}{1 + x^2} \) and

that \( |f''(x)| \leq 1 + |x| \) for all \( x \).

What can be said about the behavior of \( |f'(x)| \) for large \( |x| \)?

Note: If you are able to conjecture certain properties of \( f'(x) \) but not able to prove them, state the conjectures carefully and completely.

If you find examples that add information, give them.

Problem 2. Suppose \( \theta \) is positive and irrational and that \( f \) is a continuous periodic function with period one (i.e. \( f(x+1) = f(x) \) for all \( x \)).

Consider the averages:

\[
\sigma_N(f) = \frac{1}{N} \sum_{0}^{N-1} f(n\theta) \quad (N = 1, 2, \ldots).
\]

Show that \( \lim_{N \to \infty} \sigma_N(f) \) exists. What is it?

Hint: Try it first for \( e^{2\pi ipx} \), \( p \) integral.
Problem 3. Suppose $a(t)$ and $b(t)$ are continuous on $[0, \infty)$ and that
\[
\int_0^\infty |b(s)| \, ds < \infty.
\]

Suppose all functions $y(t)$ satisfying $\frac{dy}{dt}(t) = a(t) \, y(t)$ for $t \geq 0$ are bounded.

Does it follow that all functions satisfying $\frac{dy}{dt}(t) = a(t) \, y(t) + b(t)$ for $t \geq 0$ are bounded?

Give proof or counterexample (or conjecture and argument).

Problem 4. Suppose $f$ is the conformal map of the unit disk onto the wedge $|\theta| < \pi/4$ which carries $-1$ to $0$, $0$ to $1$ and $1$ to $\infty$.

Find $f(\frac{4}{5})$.

Problem 5. There are at least two ways in which the improper integral
\[
\int_0^{2\pi} \frac{d\theta}{1 - e^{-i\theta}}
\]
can be interpreted:

1. \[
\lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{2\pi - \varepsilon} \frac{d\theta}{1 - e^{-i\theta}}
\]
2. \[
\lim_{r \nearrow 1} \int_0^{2\pi} \frac{d\theta}{1 - re^{-i\theta}}
\]

Surprise! They're different.

Compute these two limits. Can you "explain" your answer?
Problem 6. Let \( \{\phi_n\}_1^\infty \) be orthonormal in \( L^2[0,1] \).

Let \( \{a_n\} \) be a real sequence with \( \sum a_n^2 \log n < \infty \).

Let \( S_n = \sum_{k=1}^{n} a_k \phi_k \).

Prove that \( \{S_n\} \) converges a.e. on \([0,1]\).

Problem 7. Evaluate

\[
\int_{0}^{\infty} \frac{\log x}{1 + x^2} \, dx.
\]

Problem 8. Let \( f \) be a twice continuously differentiable function on some open set in the plane.

Suppose that if the disk of radius \( r \) about \( x_0 \) is in the domain of \( f \), then the average of \( f \) on the boundary of the disk is equal to \( f \) at the center of the disk.

Use the divergence theorem to show that

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.
\]
Qualifying Exam - Analysis
January 1980

Do 5 of the following 7 problems.

1) Let $r_k$ be an enumeration of the rational numbers. Prove

$$
\sum_{k=1}^{\infty} z^{-k} |x - r_k|^{-1/2}
$$

converges for almost every real number $x$.

2) Let $f(z)$ be holomorphic in $|z| < 1$. Assume

$$
|f(z)| \leq \frac{1}{(1 - |z|)^{1/2}}.
$$

Prove

$$
|f'(z)| \leq \frac{c}{(1 - |z|)^{3/2}}.
$$

3) Let

$$
f(x) = \begin{cases} 
  x \sin 1/x & 0 < x \leq 1 \\
  0 & x = 0 
\end{cases}
$$

a) Is $f(x)$ absolutely continuous on $0 \leq x \leq 1$?

b) Is $f(x)$ of bounded variation on $0 \leq x \leq 1$?
4) Let \( f(z) \) be analytic in \( 0 < |z| < 1 \), and assume \( |f(z)| \leq \frac{C}{\sqrt{|z|}} \).
Prove that there is a function \( g(z) \) such that \( g(z) = f(z) \)
for \( 0 < |z| < 1 \) and \( g(z) \) is regular for \( |z| < 1 \).

5) If \( f(x) \) and \( g(x) \) are absolutely continuous functions of a real
variable \( x \), is

\[
h(x) = f(g(x))
\]

absolutely continuous.

6) Let \( f(t) \) be in \( L'(\mathbb{R}) \). Show

\[
\lim_{x \to \infty} \int e^{ix^2} f(t) dt = 0.
\]

7) Prove

\[
\int_{|z| = 1} \frac{dz}{\sqrt{4z^2 + 4z + 3}} = 1\pi
\]

if the integral is taken in the counterclockwise sense and \( \sqrt{4x^2 + 4x + 3} \) is to be positive for \( x = 1 \).
Analysis Qualifying Examination

Do any 5 of the following 10 problems:

1) Prove
\[
\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 + 4n^2 \pi^2}.
\]

2) Show
\[
\int_{0}^{\infty} \frac{\log^2 t}{1 + t^2} \, dt = \frac{\pi^2}{8}.
\]

3) Let \( u(x, t) \) be a function of two variables such that

i) \( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + m^2 u = 0 \quad \text{for} \quad m \neq 0 \)

ii) \( u_0(x) = u(x, 0) \) is in \( L^2 \)

iii) \( u_1(x) = (\frac{\partial u}{\partial t})(x, 0) = 0. \)

Prove \( u(x, t) \to 0 \) as \( t \to \infty \).

4) Suppose \( a(t) \) is a function such that any solution of the differential equation

\[
u'(t) + a(t)v(t) = 0
\]

is bounded as \( t \to +\infty \). Is every solution of the equation

\[
u'(t) + a(t)v(t) = b(t)
\]

bounded as \( t \to +\infty \) for every \( b(t) \) in \( L^1 \)?
5) Suppose $f$ is analytic in the whole complex plane and for some $\rho > 0$
\[ |f(z)| \leq e^{\frac{1}{2} \rho}. \]

Suppose
\[ \lim_{\log r \to \infty} \frac{\log \log \max_{|z|=r} |f(z)|}{\log r} \]

is not an integer. Show $f(z) = 0$ for some value of $z$.

6) $F(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}, \quad 0 < \alpha < 1$

is a function holomorphic for $|z| < 1$. Show that $F(z)$ can be analytically continued across some arc of the unit circle.

Hint: $\frac{1}{n^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-nt} t^{\alpha-1} \frac{dt}{t}$.

7) Let $f(z)$ be holomorphic in $|z| < 1$ and continuous in $|z| \leq 1$.
Let $g(z)$ be holomorphic in $|z| > 1$ and continuous for $|z| \geq 1$.
Suppose $f(z) = g(z)$ if $|z| = 1$. Prove $g(z)$ is an analytic continuation of $f(z)$.

8) Let $f_n$ and $f$ be $L^1$ functions on the unit interval, $[0, 1]$.
Suppose
\[ f_n \to f \text{ a.e.,} \]

and
\[ \int_0^1 |f_n| dx \to \int_0^1 |f(x)| dx. \]

Prove
\[ \int_0^1 |f_n(x) - f(x)| dx \to 0. \]
9) Let $V$ be a subspace of $L^2[0,1]$ such that

$$\sup_{x \in [0,1]} |f(x)| \leq C\left(\int_0^1 |f(x)|^2 \, dx\right)^{1/2}$$

for some constant $C$. Show $V$ is finite dimensional.

10) Let $f(x) \in L^1[-\infty,\infty]$. Show that the function

$$t \mapsto \int_{-\infty}^{\infty} e^{ixt} f(x) \, dx$$

is continuous.
Qualifying Exam in Analysis
August, 1981

Do any 4 of the following problems.

1. Show that there is one and only one function \( f \) with the following properties:
   \( f \) is holomorphic in a region containing \( \{ z : |z| \leq 1 \} \), \( f \) has a simple zero at \( z = \frac{1}{2} \), \( f \) has a double zero at \( z = \frac{(1+i)}{2} \), and \( f(0) = \frac{1}{4} \).

2. Let \( C_0 \) be the set of continuous function on \( \mathbb{R}^2 \) that vanish off some compact set. Let \( L(f) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{s} t^2 f(s, t) \, dt \right] \, ds \), for \( f \in C_0 \).
   Then there is a Borel measure \( \mu \) on \( \mathbb{R}^2 \) such that \( L(f) = \int f \, d\mu \).
   
   a) Find \( \mu(E) \) where \( E = (0, 1) \times (0, 1) \).
   
   b) Find the Lebesgue decomposition of two dimensional Lebesgue measure \( \mu \) with respect to \( \mu \).

3. Show that the polynomials \( P_n(x) = \frac{d^n}{dx^n} \left[ (x^2-1)^n \right] \), \( n = 0, 1, 2, \ldots \), suitably normalized, form an orthonormal basis for \( L^2((-1, 1)) \).
4. a) Suppose $f$ is an entire function and there are positive constants $A, B, C$ such that $|f(z)| \leq A|z|^B + C$, for all $z \in \mathbb{C}$. Show that $f$ is a polynomial.

b) Suppose $f$ is an entire function and there are polynomials $p_0(z), \ldots, p_n(z)$ with $p_n(z) \neq 0$ such that

$$
\sum_{k=0}^{n} p_k(z) [f(z)]^k = 0.
$$

Show that $f$ is a polynomial.

5. Let $\gamma : [0, 1] \to \mathbb{C}$ be a smooth curve such that $\gamma(0) = 0$, $\gamma(1) = 1$ and $\gamma(t) \neq \pm 1$ for $0 \leq t \leq 1$. Show that there is an integer $k$ such that

$$
\int_{\gamma} \frac{dz}{1+z^2} = \frac{\pi}{4} + k\pi.
$$

6. Suppose that $f$ is a non-negative measurable function on the real line such that $\int_{-\infty}^{\infty} f(x) \, dx < \infty$. Show that for every $\varepsilon > 0 \exists \delta > 0$ such that if $E \subseteq \mathbb{R}$ is measurable and $\int_{E} 1 \, dx < \delta$ then $\int_{E} f(x) \, dx < \varepsilon$.

7. Compute the maximum of $|\int_{-1}^{1} f(x) e^x \, dx|$ where $f$ ranges over all measurable functions on $(-1, 1)$ such that $\int_{-1}^{1} |f(x)|^2 \, dx \leq 1$, $\int_{-1}^{1} f(x) \, dx = 0$, and $\int_{-1}^{1} f(x) x \, dx = 0$. 
Do any 5 of the following problems.

1. Let \((X, \mathcal{M}, \mu)\) be a measure space, \(g\) a nonnegative Borel measurable function on \(X\). Define a measure \(\nu\) on \(\mathcal{M}\) by

\[
\nu(E) = \int_E g \, d\mu \quad (E \in \mathcal{M}).
\]

Show that if \(f\) is a Borel measurable function on \(X\), then

\[
\int_X f \, d\nu = \int_X fg \, d\mu,
\]

in the sense that if one of the integrals exists, then so does the other, and the two integrals are equal. (Intuitively: \(\frac{d\nu}{d\mu} = g\), so \(d\nu = g \, d\mu\).

2. Find all entire functions \(f\) such that \(|f(z)| = 1\) whenever \(|z| = 1\).

3. Let \(m\) denote Lebesgue measure on \([0,1]\). Suppose that the sequence \(\{f_n\}\) \((n = 1, 2, \ldots)\) and \(f\) satisfy:

\(f_n \in L^1(m)\) for each \(n\), \(f \in L^1(m)\),

\[
\lim_{n \to \infty} f_n(x) = f(x) \quad \text{a.e. on } [0,1],
\]

and

\[
\lim_{n \to \infty} \int_0^1 |f_n| \, dm = \int_0^1 |f| \, dm.
\]

Does it follow that \(f_n \to f\) in \(L^1(m)\)?
4. Let $U$ be the open unit disc. How many fixed points can a holomorphic $f : U \to U$ have without being the identity map?

5. Let $f_n : (0,1) \to \mathbb{R}$ $(n = 1, 2, \ldots)$ be differentiable functions such that

$$\lim_{n \to \infty} f_n(x) = f(x)$$

and

$$\lim_{n \to \infty} f'_n(x) = g(x)$$

for all $x$. Show that if the $f'_n$ and $g$ are continuous and

$$\sum_{n=1}^{\infty} \int_{0}^{1} |f'_{n+1} - f'_n| \, dm < \infty,$$

then $f'(x) = g(x)$. ( $m$ denotes Lebesgue measure.)

6. Given any proper open subset $\Omega$ of the complex plane, show that there is a holomorphic function $f \in H(\Omega)$ which cannot be analytically continued to any larger open set.
7. Denote \( S = \{0, 1, \ldots\} \). Let \( A = \{ \alpha(x, y); x, y \in S \} \) be a symmetric strictly positive matrix, and define

\[
L^2(A) = \{ f : S \times S \to \mathbb{R} : \|f\|_{L^2(A)} < \infty \},
\]

where

\[
\|f\|_{L^2(A)} = \left( \sum_{x, y \in S} \alpha(x, y) f^2(x, y) \right)^{1/2}.
\]

Given \( f : S \to \mathbb{R} \), define \( \nabla f : S \times S \to \mathbb{R} \) by

\[
\nabla f(x, y) = f(x) - f(y).
\]

Introduce the norm

\[
\Phi(f) = \|\nabla f\|_{L^2(A)}
\]

and the space

\[
H = \{ f : \Phi(f) < \infty \}.
\]

Show that \( H \) is a Hilbert space with norm \( \Phi \).

8. Suppose that \( f \) is entire, and real only on the real axis. Argue that \( f \) is linear.
Qualifying Exam

ANALYSIS

August 24, 1982

Do **ALL** Problems.

1. Let $\mu$ be a positive $\sigma$ finite measure. Suppose $1 < p < q < \infty$ and that $M$ is a subspace of $L^p(\mu)$. Suppose further that there is a constant $C$ such that

$$\left( \int |f|^q \, d\mu \right)^{1/q} \leq C \left( \int |f|^p \, d\mu \right)^{1/p}$$

for all $f \in M$. Show that for each $F \in L^q(\mu)$ there is a $G \in L^p(\mu)$ such that

$$\left( \int |G|^p \, d\mu \right)^{1/p'} \leq C \left( \int |F|^q \, d\mu \right)^{1/q'}$$

and $\int F \, d\mu = \int G \, d\mu$ for all $f \in M$. Here $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

2. Let $\{a_n\}$ be a sequence of complex numbers such that $\lim_{n \to \infty} na_n = 0$. Suppose that

$$\lim_{r \to 1, \, 0<r<1} \sum_{k=0}^{\infty} a_k r^k = L.$$ 

Show that

$$\lim_{n \to \infty} \sum_{k=1}^{n} a_k = L.$$ 

(Hint: Compare $\sum_{k=1}^{n} a_k$ with $\sum_{k=1}^{\infty} a_k \left(1 - \frac{1}{n}\right)$.)
3. Show that if $\alpha > 0$, the function $F(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$ defined originally for $|z| < 1$ has an analytic continuation into the complex plane cut from 1 to $\infty$ along the real axis. Hint: Consider the integral

$$\int_{0}^{\infty} e^{-nx} x^{\alpha-1} \, dx.$$

4. Evaluate:

$$\lim_{T \to \infty} \int_{\gamma_T} e^{-z^2} \, dz,$$

where $\gamma_T(t) = te^{i\pi/4}$, $0 \leq t \leq T$.

5. Suppose $f$ is holomorphic in $H = \{z : \text{Im } z > 0\}$, $f(H) \subseteq H$ and $f(i) = i + 1$. Find a sharp upper bound for $|f(2i)|$. 
Qualifying Exam

ANALYSIS

January 18, 1983

Do as many problems as possible.

1. \( f(t) \) decreases monotonically to 0 as \( t \to \infty \) and is in the class \( C^\infty \).

   Show

   \[
   \lim_{N \to \infty} \int_0^N e^{itx} f(t) \, dt \text{ exists for } x \neq 0.
   \]

2. Evaluate \( \int_0^\infty \frac{x-a}{1+x} \, dx \) for \( 0 < a < 1 \).
3. Let \((X, \mathcal{M}, \mu)\) be a \(\sigma\)-finite measure space. For every measurable \(f : X \to \mathbb{C}\) and \(1 < p < \infty\) let \(|f|_p\) be the least number in \([0, \infty)\) for which

\[
\int_X |f| \, d\mu \leq |f|_p(u(x))^{\frac{1}{q}} \quad \text{for all} \quad x \in \mathcal{M},
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\). Let \(M_p\) be the set of measurable \(f\) such that \(|f|_p < \infty\).

Prove:

a) \(L_p(\mu) \subseteq M_p\).

b) \(\mu\{x : |f(x)| > \lambda\} < \frac{|f|_p^p}{\lambda^p}\) for \(\lambda > 0\).

c) If \(\{f_n\}\) is a sequence of measurable functions then

\[
\liminf_{n \to \infty} |f_n|_p \leq \liminf_{n \to \infty} |f_n|_p.
\]

d) If \(X = \mathbb{R}\) and \(\mu\) is Lebesgue measure then for \(f \in L_1\) and \(g \in M_p\) it follows that \(f*g \in M_p\) and \(|f*g|_p \leq |f|_1 |g|_p\).
4. For each of the following determine if there is a function \( f(z) \), holomorphic in the open unit disc, which satisfies the given property. Either give an example or a proof.

a) \( f\left(\frac{1}{2}\right) = \frac{1}{n} \) for \( n = 2, 3, 4, \ldots \)

b) \( f \) bounded, \( f(0) = 1 \) and \( f(1 - \frac{1}{n^2}) = 0 \) for \( n = 2, 3, 4, \ldots \)

c) \( f^{(n)}(0) = \frac{n^n}{2^n} \) for \( n = 1, 2, 3, \ldots \)

5. Suppose \( f \) and \( g \) are real valued measurable functions on \( \mathbb{R} \) which are periodic, with period \( 1 \). If \( g \) is bounded and

\[
\int_0^1 |f| < \infty \quad \text{then show}
\]

\[
\lim_{n \to \infty} \int_0^1 f(x) g(nx) \, dx = \int_0^1 f(x) \, dx \int_0^1 g(x) \, dx .
\]

Hint: First assume \( f \) is continuous.

6. Let \( f \) be holomorphic in the unit disc and \( |f| \leq M \). If \( |f(0)| = a > 0 \) and \( n \) is the number of zeroes of \( f \) in the disc of radius \( 1/3 \) then show

\[
n \leq \frac{1}{\log 2} \log \left( \frac{M}{a} \right) .
\]

Hint: Consider \( g(z) = f(z) \prod_{k=1}^{n} \left( \frac{z_k}{z_k - z} \right) \) where \( z_1, z_2, \ldots, z_n \) are the zeroes of \( f \) in the disc of radius \( 1/3 \).
ANALYSIS Qualifying Exam
August 23, 1983

Give complete solutions to any six of the following problems. You may appeal to standard theorems as needed.

1. Let \( \Lambda \) be a bounded linear operator from the Banach space \( X \) onto the Banach space \( Y \). Show that there is a positive number \( k \) such that for each \( y \in Y \) there is an \( x \in X \) with \( y = \Lambda x \) and \( \|x\| \leq k\|y\| \).

2. Give an example of a sequence of functions \( f_n \) on \([0,1]\) such that each \( f_n \) is Riemann integrable, \( |f_n| \leq 1 \) for all \( n \), \( f_n \to f \) everywhere, but \( f \) is not Riemann integrable.

3. Suppose

\[
 f(z) = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}},
\]

and let \( D_R \) denote the disc with center \( 0 \) and radius \( R \). Compute

\[
 \int_{D_R} |f'|^2 \, dm, \quad 0 < R < 1,
\]

where \( m \) is 2-dimensional Lebesgue measure.
4. Let $\mu$ be a (positive) measure on $\mathbb{R}^1$, with $\mu(\mathbb{R}^1) < \infty$. Define

$$F(x) = \mu((\infty, x]) \quad \text{for real } x.$$ 

Evaluate:

$$\int_{-\infty}^{\infty} (F(x+c) - F(x)) \, dx \quad \text{for real } c.$$

5. Show that $\{\varphi_n; n \geq 1\}$ is a complete orthonormal set for $L^2[0,1]$ (Lebesgue measure) if and only if

$$\sum_{n=1}^{\infty} \left[ \int_{0}^{x} \varphi_n(t) \, dt \right]^2 = x$$

for all $x \in [0,1]$.

6. Let $\mu$ be a probability measure on $\mathbb{R}^1$, i.e. $\mu(\mathbb{R}^1) = 1$. Write

$$\psi(u) = \int_{-\infty}^{\infty} e^{iux} \, d\mu(x) \quad \text{for complex } u.$$

Suppose $|\psi(u)| = 1$ for some $u \neq 0$. What can one conclude about $\mu$?
7. Let \( Q \) be the square with vertices at \( 1, i, -1, -i \).
Show that there is a conformal map \( f \) of the unit disc onto
\( Q \) that fixes each of the points \( 1, i, -1, -i \). By comparing
\( f(iz) \) and \( f(z) \), prove a precise statement to the effect
that \( 3/4 \) of the coefficients in the expansion
\[
f(z) = \sum a_n z^n
\]
are 0.

8. An integer sequence \( A_n \) is determined inductively by the
convolution equation
\[
A_n = \sum_{k=1}^{n-1} \binom{n}{k} A_k A_{n-k}, \quad n \geq 2,
\]
\( A_1 = 1. \)
Show that \( A_n \leq 4^n n! \).
1. Evaluate by contour integration.

\[ \int_{0}^{\infty} \frac{\sin x}{x(x^2+4)} \, dx. \]

2. Let \( f \) be a non-constant meromorphic function defined on \( \mathbb{C} \), such that \( f(z+1) = f(z) \) and \( f(z+i) = f(z) \) for all \( z \in \mathbb{C} \). Prove that there is no \( \alpha \in \mathbb{C} \) such that

a) \( f \) has a simple pole at \( \alpha \) (and hence at \( \alpha + m + ni, \ m, n \in \mathbb{Z} \)) and

b) \( f \) has no poles other than at \( \alpha + m + ni, \ m, n \in \mathbb{Z} \).

3. Let \( \lambda > 1 \). Show that \( \lambda - z - e^{-z} = 0 \) has exactly one solution in the half plane \( \text{Re } z > 0 \), and that it is real. What happens to this solution as \( \lambda \to 1 \) ?
4. For \( x, y \in \mathbb{R} \) and \( u > 0 \) define

\[
f_u(x, y) = (2\pi u)^{-1/2} \exp\left\{ -\frac{|y-x|^2}{2u} \right\} .
\]

Let \( \varphi \) be a continuous function with compact support on \( \mathbb{R} \).

Show that for any \( x \neq 0 \),

\[
\lim_{t \to \infty} \int_0^1 \int_0^R f_u(x, y) \ t \varphi(ty) \ dy \ du
\]

\[
= \int_0^1 f_u(x, 0) \ du \int_0^R \varphi(y) \ dy .
\]

5. Define \( h_s(t), \ 0 \leq s \leq \pi \), on \([-\pi, \pi]\) by

\[
h_s(t) = \begin{cases} 
  t & |t| \leq s \\
  -s & t \leq -s \\
  s & t \geq s 
\end{cases}.
\]

Use these functions to check that

\[
\min(s, t) = \frac{st}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k^2},
\]

\( 0 \leq s, t \leq \pi \).
6. Suppose that \( f \in L^1_{\text{loc}}(\mathbb{R}) \) and that for every interval \( I \) we have

\[
\lim_{h \to 0} \int_I \frac{|f(x+h) - f(x)|}{h} \, dx = 0.
\]

Show that there is a constant \( c \) such that \( f = c \) almost everywhere.

7. Let \( H \) be a complex Hilbert space and let \( T : H \to H \) be a bounded linear operator. Suppose that \( \langle x, y \rangle = 0 \) implies that \( \langle Tx, Ty \rangle = 0 \). Show that there is a constant \( c > 0 \) such that \( \| Tx \| = c \| x \| \) for all \( x \in H \).

(Hint: use an orthonormal basis.)
1. Let \( \mathcal{U} \) be an open set in the complex plane, and \( f(z) \) a function holomorphic in \( \mathcal{U} \), but not a polynomial. Prove that there is a point \( z_0 \) in \( \mathcal{U} \) such that the Taylor series for \( f \) at \( z_0 \) is a power series in \( (z-z_0) \) which does not have any zero coefficients.

2. Let \( f, g, \) and \( h \) be real valued functions defined for \( -\infty < t < \infty \), one of which has a graph that is a closed set. Suppose that \( f(x+y) = g(x) + h(y) \) for all \( x \) and \( y \). Find all possible choices for these functions.

3. Justify the formula: \[ \arctan(z) = \frac{1}{2i} \log \left( \frac{1 + iz}{1 - iz} \right) \]

and discuss its range of validity and domain of definition.

4. Let \( \mu \) be Lebesgue measure on \( [0, \infty) \) and \( f \in L^1(\mu) \).

(a) Show that it is not necessarily true that \( \lim_{x \to \infty} f(x) = 0 \) off a set of zero measure.

(b) Show that for any \( \varepsilon > 0 \), there is a set \( E \subset [0, \infty) \) such that \( \mu(E) < \varepsilon \), and \( \lim_{x \to \infty} f(x) = 0 \) \( x \notin E \).

(c) Show that there exist points \( x \) such that \( \lim_{n \to \infty} f(x+n) = 0 \).
5. Let $c_n > 0$, and $\lim_{n \to \infty} \frac{1}{N} \sum_{k=1}^{n} c_k = 0$. Show that
\[
\lim_{n \to \infty} \frac{1}{N} \sum_{k=1}^{n} (c_k)^{1/p} = 0 \quad \text{for every } p, \ 1 \leq p < \infty,
\]
but that this assertion cannot necessarily be made for $p = 1/2$ or $3/4$, or in fact for any $p, \ 0 < p < 1$.

6. Consider the class of all functions $f(z)$, holomorphic in the upper half plane, and such that $f(i) = 0$ and such that $\{f(z)\}^2$ is never a real number $\geq 4$. How large can $|f'(i)|$ be?

7. Let $\mathcal{A}$ be the class of all real non-negative continuous functions $f$ defined on $[-1,1]$ and such that
\[
\int_{-1}^{1} f(x)dx = 1 + \int_{-1}^{0} f(x)dx
\]
Let $\|f\| = \max_{-1 \leq x \leq 1} |f(x)|$.

Find $\inf_{f \in \mathcal{A}} \|f\|$, and show that this value cannot be attained by any $f \in \mathcal{A}$.

8. Let $\mu$ be Lebesgue measure in $\mathbb{R}^3$, and $A_1, A_2, A_3$ be bounded open sets with $\bigcap_i A_i = \emptyset$. Prove that
\[
\mu\left(\bigcup_i A_i\right) \geq \frac{3}{2} \min_i \mu(A_i)
\]
Can the number $3/2$ be increased?
1. Let $f \in L^1(\mathbb{R})$, $f \geq 0$.

Suppose $E_n \subset \mathbb{R}$ are measurable sets, and

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \emptyset.$$

Prove

$$\lim_{n \to \infty} \int_{E_n} f(x) \, dx = 0.$$

(Here $|A|$ denotes the Lebesgue measure of $A$.)

2. In each of the following, either give an example of a function holomorphic in $U = \{|z| < 1\}$ which satisfies the required conditions, or prove that no such function exists.

(a) $f(0) \neq 0$, $f(1 - \frac{1}{n^2}) = 0$ $n = 2, 3, 4, \ldots$

(b) $|f(z)| \leq 1$ for all $z \in U$, $f''(0) = 3$

(c) $f^{(n)}(0) = n!$ $n = 0, 1, 2, \ldots$

(d) $\lim_{r \to 1} \Re f(re^{i\theta}) = 0$ for $0 < \theta < \pi$, $f(0) = 1/2$.

(e) $f(0) = 1$, $f(1/2) = 4$, and $\Re f(z) > 0$ for $z \in U$. 
3. Let \( f \) be continuous on \( \mathbb{R} \) and \( \lim_{x \to \infty} f(x) = L \). Show \( \lim_{y \to 0^+} \int_0^\infty e^{-xy} f(x) \, dx = L \).

4. If \( f \) is an entire function such that for all \( x \) and \( y \),

\[ |f(x+iy)| \leq e^{xy} \] and \( f(0) = 1 \), find \( f \).

5. If \( f \in L'(\mathbb{R}) \) show that

\[ \lim_{\lambda \to \infty} \int_{-\infty}^{\infty} e^{i\lambda t^2} f(t) \, dt = 0. \]
6. For \( m = 1, 2, 3, \cdots \), define
\[
f_m(z) = \frac{z}{1+z^m}.
\]

(a) Show that \( f_m \) is one-to-one in the open unit disc \( U \) if and only if \( m \) is 1 or 2.

(b) Find the region \( f_2(U) \).

7. If \( f : [0,1] \to (0,\infty) \) is measurable, must
\[
\lim_{n \to \infty} \int_0^1 \frac{f^n(x)}{1+f^{2n}(x)} \, dx
\]
exist?

When it exists, what is it?
8. Let \( f \) and \( g \) be measurable functions on \( \mathbb{R} \) of period one with

\[
\int_0^1 |f|^2 < \infty, \quad \int_0^1 |g|^2 < \infty.
\]

Find

\[
\lim_{k \to \infty} \int_0^1 f(kx)g(x) \, dx.
\]

9. Let \( M = \{ \text{All } F : \mathbb{R} \to [0,1] \text{, such that } F \text{ is non-decreasing and right continuous} \} \). Let \( F_n \in M \), \( n = 1, 2, 3, \cdots \). Show that there is a subsequence \( \{F_{n_k}\} \) and an element \( F \in M \) such that \( \lim_{k \to \infty} F_{n_k}(x) = F(x) \) at all points \( x \) where \( F \) is continuous.

10. Find the residue at \( z = 0 \) of the function

\[
f(z) = \frac{e^{1/z}}{z^2(1+z)^2}.
\]
1. Suppose that $u$ is harmonic in the complex plane and that
\[ \lim_{|z| \to \infty} \frac{|u(z)|}{|z|} = 0. \] Show that $u$ is constant.

2. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in the closed unit disc except at $z_0$, $|z_0| = 1$, where $f$ has a simple pole.
Show that $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = z_0$.

3. Let $\psi$ be a continuous function on $\mathbb{R}$ with
\[ \int_{-\infty}^{\infty} |\psi(t)| \, dt < \infty \]
and \[ \int_{-\infty}^{\infty} \psi(t) \, dt = 0. \] Show that $\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \psi(t/\epsilon) \, g(t) \, dt = 0$
for every $g$ which is continuous on $\mathbb{R}$ with compact support.

4. Let $\mu$ be a positive measure. $M \subseteq L^1(\mu)$ is a closed subspace such that $M \subseteq \bigcup_{p>1} L^p(\mu)$. Show that there is a $p>1$ such that $M \subseteq L^p(\mu)$.
5. a) Suppose \( f \in L^\infty(-\pi, \pi) \) with Fourier series
\[
f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(k) e^{ikx}.
\]
Suppose that \( \hat{f}(k) \geq 0 \) for all \( k \).
Show that \( \sum_{k=-\infty}^{\infty} \hat{f}(k) \) converges. (Hint: Consider the Poisson integral of \( f \).)

b) What if \( f \in L^2(-\pi, \pi) \)?

6. Let \( \{f_n\}_{n=1}^{\infty} \) be an orthonormal set in \( L^2(d\mu) \); \( \mu \) is a positive measure. Show that
\[
\lim_{k \to \infty} \frac{f_1 + f_2 + \cdots + f_{n_k}}{\sqrt{n_{k+1}}} = 0 \quad \text{a.e. } d\mu.
\]
Here \( n_k = 2^{k^2} \).
(Hint: To get started, prove the inequality
\[
\mu(\{x : |F(x)| > \varepsilon\}) \leq \frac{1}{\varepsilon^2} \int |F|^2 \, d\mu \quad \text{valid for any } F \in L^2(d\mu).
\]
7. Let $L_1,L_2$ be a pair of straight lines in the plane that intersect at angle $\alpha$. Let $u$ be a harmonic function defined on $\mathbb{C}$ such that $u \equiv 0$ on $L_1 \cup L_2$.

a) Assuming that $\alpha$ is not a rational multiple of $\pi$, show that $u \equiv 0$.

b) Show that if $\alpha$ is a rational multiple of $\pi$ then there is such a $u$ with $u \not\equiv 0$ in $\mathbb{C}$.

8. Let $Y$ be the set of real valued non-negative measurable functions $f$ on $(-\infty,\infty)$ such that

$$
\int_{-\infty}^{\infty} f^2(x) dx = 2, \quad \int_{-\infty}^{\infty} f^5(x) dx = 5.
$$

a) Calculate $m = \sup \int_{-\infty}^{\infty} f^3(x) dx$.

b) Is there $f \in Y$ such that $\int_{-\infty}^{\infty} f^3(x) dx = m$?
9. Find a sequence \( \{c_n\}_{n=1}^{\infty} \) so that the integrals

\[
c_n \int_{\mathbb{R}^2} f(x-y)g(ny) \, dm(y)
\]

converge to a limit as \( n \to \infty \), provided that \( f \) and \( g \) satisfy some reasonable conditions.

(Here \( m \) denotes Lebesgue measure in \( \mathbb{R}^2 \)). State such conditions, identify the limit and prove the resulting theorem.
1. Let \( \mathcal{S} \) be a countable collection of nonnegative continuous functions \( f(x) \) defined for \( 0 \leq x < \infty \). Show that there exists a continuous function \( \phi(x) \) with the property that for each \( f \in \mathcal{S} \), there is a choice of \( c \) such that \( f(x) < \phi(x) \) for all \( x > c \). Then discuss the situation when \( \mathcal{S} \) is noncountable.

2. The improper integral \( \int_0^{2\pi} \frac{d\theta}{1 - e^{i\theta}} \) was interpreted in three different ways by Tom, Dick, and Harry:

- **T:** \( \lim_{\delta \to 0} \int_\delta^{2\pi-\delta} \frac{d\theta}{1 - e^{i\theta}} \)
- **D:** \( \lim_{r \uparrow 1} \int_0^{2\pi} \frac{d\theta}{1 - re^{i\theta}} \)
- **H:** \( \lim_{r \downarrow 1} \int_0^{2\pi} \frac{d\theta}{1 - re^{i\theta}} \)

Calculate these three limits; is there a good explanation for the discrepancies?

3. Suppose \( f(z) \) is holomorphic for \( |z| < 1 \), and \( f(0) = 0 \). Where is the function defined by \( g(z) = f(z) + f(z^2) + f(z^3) + \ldots \) holomorphic?

4. Suppose that \( f \) is a real valued measurable function defined on \( [0, \infty) \) with compact support. Let \( g(\beta) = \operatorname{meas} \{ \text{all } x \text{ with } |f(x)| > \beta \} \) and suppose that there is \( C > 0 \) with \( g(\beta) \leq C/\beta \). Show that if \( 0 < p < 1 \) then \( \int_0^\infty |f(x)|^p \, dx < \infty \).

5. Let \( f \) and \( g \) be continuous complex valued functions defined on \( -\infty < x < \infty \), and let \( f \) be periodic with period \( \alpha \) and \( g \) periodic with period \( \beta \). Determine the precise conditions under which \( f + g \) is a periodic function.
6. Find a conformal mapping that takes the region \( D \) onto the upper half plane, with 0 going to 1 and 1 to \( \infty \).

7. Let \( \{ \phi_n \} \) be an orthonormal basis for \( L^2[-1,1] \), and let \( s_n = \sum a_k \phi_k \) where \( \sum |a_k|^2 < \infty \).

(a) Show that there is a function \( f \in L^2 \) such that \( \frac{1}{\sqrt{n}} \int_{-1}^{1} |f - s_n|^2 \) converges.

(b) Must \( s_n(x) \) converge almost everywhere to \( f(x) \)?

8. Let \( E \) be a set of real numbers whose Lebesgue measure is finite and positive. Let \( A \) be the set

\[
\left\{ \text{all } x - y \text{, where } x \in E \text{ and } y \in E \right\}
\]

Prove that \( A \) contains a non-empty neighborhood of 0.

9. Define a function \( H(r) \) by \( \sum_{n=1}^{\infty} \frac{1}{4n^2 - r^2} \)

Find the exact value of \( H(5) \).
1. Evaluate:

\[
\int_{-\infty}^{\infty} \frac{\cos(x)}{\cosh(x)} \, dx
\]

here \( \cosh(x) = \frac{e^x + e^{-x}}{2} \),

2. Define \( T : C[0,1] \rightarrow C[0,1] \) by \( Tf(x) = \int_0^x f(t) \, dt \).

( \( C[0,1] \) is the space of continuous complex valued functions defined on \([0,1]\)).

a) What is the adjoint \( T^* \) of \( T \)?

b) If \( \mu \) is the unit point mass located at \( 1/2 \), what is \( T^*(\mu) \)?

3. Suppose that the series \( \sum_{n=0}^{\infty} a_n z^n \) converges when \( z = 1 \).

Prove the strongest theorem you can about regions in which this series converges uniformly.
4. A function $f$, holomorphic for $|z| < 1$, is said to belong to $B$ if $|f'(z)| \leq \frac{C}{1-|z|}$ for some constant $C$, all $|z| < 1$.

a) Suppose that $\sum_{k=0}^{N} k|a_k| \leq A \cdot N$ for all $N$ and some constant $A$. Show that $f(z) = \sum_{k=0}^{\infty} a_k z^k \in B$.

b) Show that if $|b_k| \leq A$ for all $k = 0, 1, 2, \ldots$ and some constant $A$, then $f(z) = \sum_{k=0}^{\infty} b_k z^k \in B$.

5. Suppose that $f : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic, $f(0) = 0$, $f(1) = 1$, and $f(2) = 3$. Show that $f$ cannot be one-to-one on $\mathbb{C}$. 
6. Show that there is a sequence \( \{f_n\} \) of entire functions such that

(i) no \( f_n \) is 0 at any point of \( \mathbb{C} \),

(ii) \( f(z) = \lim_{n \to \infty} f_n(z) \) exists for every \( z \in \mathbb{C} \),

(iii) \( f(0) = 0 \), \( f\left(\frac{1}{2}\right) = 1 \).

Show also that no such sequence converges uniformly on the unit disc.

7. State the Radon-Nikodym theorem and use it to prove that to every bounded linear functional \( \Lambda \) on \( L^1([0,1]) \) corresponds a \( g \in L^\infty([0,1]) \) so that the representation

\[
\Lambda f = \int_0^1 f(x) \, g(x) \, dx
\]

holds for every \( f \in L^1([0,1]) \).
8. Suppose that $f$ is continuous on $\mathbb{R}$, $f(x) > 0$ if $0 < x < 1$ and $f(x) = 0$ otherwise. Let $h_c(x) = \sup \{n^c f(nx) : n = 1, 2, 3, \ldots\}$. Show that

a) $h_c \in L^1(\mathbb{R})$ if $0 < c < 1$.

b) $h_1 \notin L^1(\mathbb{R})$ but there is a constant $C$ so that $m(\{x : h_1(x) > t\}) \leq \frac{C}{t}$, for all $t > 0$. (Here $m$ denotes Lebesgue measure.)
1. Compute

\[ \int_{-\infty}^{\infty} \frac{dx}{x^3 + 8i} \]

2. Why is it not possible to have

\[ \int_X f \, d\mu = 1, \quad \int_X f^2 \, d\mu = 2, \quad \int_X f^3 \, d\mu = 3 \]

with \( \mu \) a positive measure and \( f \) a positive function on some set \( X \)?

For which positive \( f \) is it true that

\[ \int_X f \, d\mu = 1, \quad \int_X f^2 \, d\mu = 2, \quad \int_X f^3 \, d\mu = 4 \]
3. Suppose \( 1 < t_1 < t_2 < \cdots , \quad t_j \to \infty \text{ as } j \to \infty , \)

and

\[
f_n(z) = \prod_{j=1}^{n} \frac{t_j - z}{t_j + z} \quad (n = 1, 2, 3, \cdots) .
\]

Prove that

\[
f(z) = \lim_{n \to \infty} f_n(z)
\]

exists for all \( z \) in the right half-plane \( P = \{ z : \text{Re } z > 0 \} \),

and that either

(a) \( f(z) = 0 \) for every \( z \) in \( P \), or

(b) \( f(z) \neq 0 \) for every \( z \) in \( P \) which is not one of the \( t_j \)'s.

For which sequences \( \{ t_j \} \) will (a) happen, and for which will (b) happen?
4. Let $X$ be the set of all holomorphic functions $f$ in the open unit disc $U$ that have $f(0) = 0$ and

$$\frac{1}{\pi} \int_U |f'|^2 \, dm < \infty$$

where $m$ is 2-dimensional Lebesgue measure.

If $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$, and both $f$ and $g$ are in $X$, calculate

$$(f, g) = \frac{1}{\pi} \int_U f' \overline{g'} \, dm$$

in terms of the coefficients $a_n$, $b_n$.

Show that $(f, g)$ is an inner product that makes $X$ a Hilbert space.
5. Suppose \( \int_{0}^{1} |f_n(x)|^2 \, dx \leq 10 \) for \( n = 1, 2, 3, \ldots \), and

\[
\lim_{n \to \infty} f_n(x) = 0
\]

for every \( x \) in \([0, 1]\).

(a) Does it follow that \( \lim_{n \to \infty} \int_{0}^{1} |f_n(x)|^2 \, dx = 0 \)?

(b) Does it follow that \( \lim_{n \to \infty} \int_{0}^{1} |f_n(x)| \, dx = 0 \)?
6. Let \( K = K_0 \cup K_1 \), where \( K_0 \) and \( K_1 \) are disjoint compact sets, as in the picture,

and let the path \( \Gamma \) surround \( K_1 \) as indicated.

Assume \( m(K) > 0 \), where \( m \) is 2-dimensional Lebesgue measure. Define

\[
f(z) = \int_{K} \frac{dm(w)}{w - z} \quad (z \in \mathcal{C}) ,
\]

Prove that \( f \) is continuous and bounded on \( \mathcal{C} \), holomorphic outside \( K \), not constant, and calculate

\[
\frac{1}{2\pi i} \int_{\Gamma} f(z) \, dz .
\]
7. Compute

\[ \lim_{n \to \infty} \int_0^\infty (1 + \frac{x}{n})^{-n} \sin(\frac{x}{n}) \, dx. \]

Prove your answer is correct.

8. Let \( f_n \) be integrable with respect to Lebesgue measure on \([-1,1]\) for \( n = 1, 2, \cdots \). Assume that

\[ \lim_{n \to \infty} \int_0^1 f_n(x) \, g(x) \, dx \]

exists for every continuous \( g \) on \([-1,1]\).

(i) Prove then that there is a Borel measure \( \mu \) on \([0,1]\) such that

\[ (*) \quad \lim_{n \to \infty} \int_0^1 f_n(x) \, g(x) \, dx = \int_{[0,1]} g \, d\mu \]

for all continuous functions \( g \).

(ii) Is it always the case that \( (*) \) holds for all bounded Borel measurable functions \( g \)?
DO ANY SIX PROBLEMS

1. J.E. Littlewood, in the context of Lebesgue measure, said: "There are three principles, roughly expressible in the following terms: Every measurable set is nearly a finite union of intervals, every measurable function is nearly continuous, and every pointwise convergent sequence of measurable functions is nearly uniformly convergent." State three theorems which support these statements, and prove one of the last two.

2. A real valued function \( f(x) \) defined on \( (-\infty, \infty) \) is said to be convex if \( f(\beta x + (1-\beta)y) \leq \beta f(x) + (1-\beta) f(y) \) for every \( x \) and \( y \), and every \( \beta, \ 0 \leq \beta \leq 1 \). Prove:
   (a) Any convex function is absolutely continuous on any compact set.
   (b) A convex function is differentiable on \( (-\infty, \infty) \) at all points except possibly a countable set.

3. Let \( f(z) \) be holomorphic for \( |z| < 1 \), and obey \( f''(0) = 0 \), and \( |f''(z)| < 1 \) there. Let \( C = f(1/2) \). Obtain as sharp an estimate for \( |C| \) as you can. \( f(0) = f'(0) = 0 \)

4. Let \( f(z) \) be holomorphic for \( |z| < 1 \). Let \( \{r_n\} \) and \( \{C_n\} \) be positive increasing sequences with \( \lim r_n = 1 \), \( \lim C_n = \infty \). Suppose that \( |f(z)| > C_n \) for all \( z \) with \( |z| = r_n \). Must \( f \) have a zero in \( |z| < 1 \)? [ Either give a counter-example or a proof. ]

5. Give a careful statement of the Hahn-Banach theorem, and explain either how it can be used in a proof of the Runge Theorem, or in a proof of the Weierstrass approximation theorem.
6. Let \( g(z) \) be an entire function, and \( A \) and \( B \) two complex numbers such that
\[
  g(z + A) = g(z) = g(z + B)
\]
for all values of \( z \).

(a) If \( A = 1 \) and \( B = i \), prove that \( g \) is a constant.

(b) What general hypothesis on \( A \) and \( B \) leads to the same conclusion?

7. Let \( \{g_n\} \) be a finite valued real measurable function on \((-\infty, \infty)\) and for each real number \( \lambda \), let
\[
  S(n, \lambda) = \{ x : g_n(x) < \lambda \}
\]
Define a function \( f \) by
\[
  f(x) = \inf_{n=1,2,...} g_n(x)
\]
Choose a positive number \( \beta \) and define a set by
\[
  A = \{ x : f(x) > \beta \}
\]

(a) Express the set \( A \) in terms of the sets \( S(n, \lambda) \)

(b) What is the topological nature of the set \( A \) if all the functions \( g_n \) are in fact continuous?

8. Let \( \Omega \) be a connected open set in the plane, and \( \{f_n\} \) a sequence of functions, each of which is holomorphic and one-to-one in \( \Omega \). Suppose that \( \{f_n\} \) converges to \( f \), uniformly on each compact subset of \( \Omega \). Prove that \( f \) is either constant on \( \Omega \), or is one-to-one on \( \Omega \), and that both cases can occur.
Problem 1 Find a continuously differentiable function
\[ f : [0, \infty) \rightarrow [0, \infty) \]
such that
\[ 0 < f(x)^2 \leq f'(x) \]
for all \( x \geq 0 \) or prove that no such function exists.

Problem 2 The function
\[ f(z) = \exp \left( \frac{1}{1 - z} \right) \]
has a Laurent expansion
\[ f(z) = \sum_{n=0}^{\infty} A_n z^{-n} \]
valid for \( |z| > 1 \). Find the following:
(1) \( A_0 \)
(2) \( \sum_{n=0}^{\infty} |A_n|^2 \)
(3) \( \sum_{n=0}^{\infty} |A_n| \)
(Justify all limit operations.)

Problem 3 Maximize
\[ \int_0^1 xf(z) \, dx \]
subject to the constraints
(1) \( f \) is measurable.
(2) \( \int_0^1 |f(x)|^2 \, dx = 1. \)
(3) \( \int_0^1 f(x) \, dx = 0. \)
Problem 4 Let $\mathcal{F}$ be the class of all functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

satisfying

$$\sum_{n=0}^{\infty} |a_n|^2 \leq 1.$$ 

For each complex number $w$ with $|w| < 1$ define

$$\mu(w) = \sup\{|f(w)| : f \in \mathcal{F}\}$$

(1) Find a more explicit expression for $\mu(w)$.

(2) Is it the case that for every $w$ with $|w| < 1$ there is an $f \in \mathcal{F}$ for which $f(w) = \mu(w)$?

Problem 5 Suppose $f_n : [0,1] \rightarrow [0,\infty)$ is a sequence of non-negative measurable functions satisfying

$$\int_0^1 f_n(x)^2 \, dx \leq \delta$$

for all $n$ and

$$\lim_{n \to \infty} f_n(x) = 0$$

for all $x \in [0,1]$. Find all positive numbers $p$ such that it follows that

$$\lim_{n \to \infty} \int_0^1 f_n(x)^p \, dx = 0$$

(and prove your answer).
Problem 6 Exhibit a measurable function \( f : [0,1] \times [0,1] \rightarrow \mathbb{R} \) such that for each \( t \in [0,1] \) both functions
\[
[0,1] \rightarrow \mathbb{R} : x \mapsto f(x,t)
\]
\[
[0,1] \rightarrow \mathbb{R} : y \mapsto f(t,y)
\]
are integrable, with both functions
\[
[0,1] \rightarrow \mathbb{R} : x \mapsto \int_0^1 f(x,y) \, dy
\]
\[
[0,1] \rightarrow \mathbb{R} : y \mapsto \int_0^1 f(x,y) \, dx
\]
integrable and
\[
\int_0^1 \int_0^1 f(x,y) \, dx \, dy \neq \int_0^1 \int_0^1 f(x,y) \, dy \, dx.
\]

Problem 7 Let \( f : \mathbb{R} \rightarrow \mathbb{C} \) be a continuous 2\( \pi \)-periodic complex-valued function with Fourier expansion
\[
f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.
\]
For \( t > 0 \) define
\[
u(t,\theta) = \sum_{n=-\infty}^{\infty} a_n e^{-n^2 t} e^{in\theta}.
\]
(1) Prove that
\[
\lim_{t \to 0} \int_0^{2\pi} |u(t,\theta) - f(\theta)|^2 \, d\theta = 0.
\]
(2) Prove that if \( f \) has two continuous derivatives then
\[
\lim_{t \to 0} \sup_{0 \leq \theta \leq 2\pi} |u(t,\theta) - f(\theta)| = 0.
\]
Problem 8 Evaluate

\[ \int_{-\infty}^{\infty} \ln(9 + x^2) \frac{dx}{1 + x^2} \]

Problem 9 Is there a function \( f = f(z) \) holomorphic in the unit disk \( |z| < 1 \) with the property that

\[ \lim_{n \to \infty} \inf_{|z|=r_n} |f(z)| = \infty \]

for some sequence of positive numbers \( r_n \) with

\[ \lim_{n \to \infty} r_n = 1? \]

Problem 10 Let \( X \) and \( Y \) be Banach spaces.

(1) Show by example that a vector subspace \( V \subset Y \) can have codimension one and fail to be a closed subset of \( Y \).

(2) Show by example that the image \( T(X) \subset Y \) of a continuous linear transformation \( T : X \to Y \) can fail to be a closed subset of \( Y \).

(3) Prove that if the image \( T(X) \subset Y \) of a continuous linear transformation \( T : X \to Y \) has codimension one, then it is a closed subset of \( Y \).
Problem 11 Let \( f(z) = u(x,y) + iv(x,y) \) (where \( z = x + iy \)) be a holomorphic function defined in a region plane \( \Omega \) with real part \( u \) and imaginary part \( v \). Suppose that the gradient of \( u \) does not vanish in \( \Omega \). Let \( \kappa(x,y) \) be the curvature of the curve \( u^{-1}(u(x,y)) \) at \((x,y)\) and let \( a = a(x,y) \) be the length of the gradient of \( u \):

\[
a = \sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2}
\]

Show that the function \( h \) given by

\[
h(x,y) = \frac{\kappa(x,y)}{a(x,y)}
\]

is harmonic.

Hint: Let \( s \rightarrow (x(s), y(s)) \) be a parameterization of a level curve \( u(x,y) = c \) (\( c \) a constant) with respect to arclength. Differentiate \( u(x(s), y(s)) \) twice to obtain an expression for \( \kappa \). Differentiate \( f'(z)^{-1} \).
REAL ANALYSIS

1. Let \( \mu_1, \mu_2, \text{ etc.} \) be a sequence of positive measures in a measurable space.

   (i) Suppose \( \mu_n(E) \leq \mu_{n+1}(E) \) for every measurable set, and let \( \mu(E) = \lim_{n \to \infty} \mu_n(E) \). Prove that \( \mu \) is a measure.

   (ii) Does the same result follow if \( \mu_n(E) \geq \mu_{n+1}(E) \) for every measurable \( E \) ?

2. Let \( \mu \) be a positive measure in a set \( X \), and \( f, 0 \leq f \leq \infty \), a measurable function there. If \( 0 < p < \infty \), and

\[ 0 < \int_X f \, d\mu < \infty, \]

find

\[ \lim_{L \to \infty} \int_X \ln(1 + \left(\frac{f}{L}\right)^p) \, d\mu. \]
3. Is there a positive function \( f(x) \) such that
\[
\int_{-\infty}^{\infty} f(x) \, dx < \infty ,
\]
while
\[
\int_{-\infty}^{\infty} f(x)^p \, dx = \infty \text{ if } p \neq 1 , \quad 0 < p < \infty ?
\]

4. Let \( \mu \) be a positive measure in a set \( X \) with \( \mu(X) = 1 \).

If \( f \) is a positive measurable function, which is larger
\[
\int_{X} f \ln f \, d\mu \quad \text{or} \quad \int_{X} f \, d\mu \cdot \int_{X} \ln f \, d\mu \quad ?
\]
5. Find a nonempty closed set in $L^2[0,1]$ that contains no element of least norm.

6. Let $f(x)$ and $g(x)$ be Lebesgue measurable functions.

Suppose

(i) $\int_{-\infty}^{\infty} |f(x)| \, dx = 1$  

(ii) $|g(x)| \leq 1$ for almost all $x$.

A. If

$$f_n(x) = n \int_{x+\frac{1}{n}} \cdots$$

prove that

$$\int_{-\infty}^{\infty} |f_n(x)| \, dx \leq 1.$$  

B. Use the Lebesgue differentiation theorem to find

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x) \ g(x) \, dx.$$
1. Let $A_1, A_2, \ldots$ be complex numbers. Suppose

\[(i) \quad \sum_{k=1}^{\infty} |A_{k+1} - A_k| < \infty\]

\[(ii) \quad \lim_{k \to \infty} A_k = 0\]

\[(iii) \quad \text{the radius of convergence of} \quad \sum_{k=1}^{\infty} A_k z^k \quad \text{is 1.}\]

A. Where does the series converge?

B. Find the collection of sets on which it converges uniformly.

2. Suppose that $f(z)$ is entire, and that in every power series

$$f(z) = \sum_{n=0}^{\infty} A_n (z-\xi)^n$$

one of the coefficients vanishes. Prove that $f(z)$ is a polynomial.
3. Find

$$\int_{0}^{2\pi} \frac{\sin \theta}{1 + i \sin \theta} \, d\theta.$$ 

4. Let $f(z) = \sum_{\lambda=0}^{\infty} A_{\lambda} z^\lambda$ be holomorphic in the open disc $|z| < 1$. Also, let the point $z = 1$ be an isolated, but not an essential singularity of $f(z)$.

(i) What, beyond not being essential, is true of the singularity if $|A_{\lambda}| < 1$?

(ii) What if $\lim_{\lambda \to \infty} A_{\lambda} = 0$?
5. Let \( f(z) \) be continuous in the closed disc \(|z| \leq 1\)
and holomorphic in the open disc \(|z| < 1\). Also, let
\[|f(z)| \geq 1 \quad \text{if} \quad |z| = 1, \quad \text{and let} \quad |f(0)| < 1.\]

A. Prove that \( f \) vanishes somewhere in the open disc
\(|z| < 1\).

B. If \(|\xi| < 1\), prove that somewhere in the open disc,
\( f \) takes the value \( \xi \).

6. Is the analytic function \( \ln(z^2 - 1) \) single-valued in the
\( z \)-plane less the closed interval \(-1 \leq x \leq 1\)? In other
words, is there a function \( g(z) \) holomorphic there and
such that \( z^2 - 1 = e^{g(z)} \)?
Qualifying Exam
ANALYSIS

Real Analysis

1. Show \( \lim_{n \to \infty} \frac{1}{n} \int_{1/n}^{1} \frac{\cos(x+\frac{1}{n}) - \cos x}{x^{3/2}} \, dx \) exists.

2. Let \( A_1, A_2, \ldots \) be positive numbers such that

\[
\sum_{\ell=1}^{\infty} A_\ell < \infty.
\]

Let \( r_1, r_2, \ldots \) be the rationals and let \( E \) be the set of \( x, -\infty < x < \infty \), for which

\[
\sum_{\ell=1}^{\infty} \frac{A_\ell^2}{|x-r_\ell|} < \infty.
\]

Show that \( E' \), the complement of \( E \), has Lebesgue measure 0.

3. Let \( A_1, A_2, \ldots \) and \( B_1, B_2, \ldots \) be sequences of positive numbers such that

(i) \( A_\ell \leq A_{\ell+1} \) and \( \lim_{\ell \to \infty} A_\ell = \infty \)

(ii) \( B_\ell > B_{\ell+1} \) and \( \lim_{\ell \to \infty} B_\ell = 0 \).

\( \sum_{\ell=1}^{\infty} A_\ell (B_\ell - B_{\ell+1}) < \infty \) then prove that \( \lim_{\ell \to \infty} A_\ell B_\ell = 0 \).

Hint: This may be proved by dominated convergence.

b) If hypothesis (i) is omitted does the conclusion of (a) still hold?
4. \( f \) is a real valued measurable function defined on a measure space \((S, \Sigma, \mu)\). For each \( y \in \mathbb{R}, \ E_y \) is a set which differs from \( f^{-1}((y,\infty)) \) by a set of measure 0. Use the \( E_y \) to construct a function \( g(x) \) with \( f(x) = g(x) \) a.e.

5. A Lebesgue integrable function \( f \) defined on \([0,4]\) has the property that

\[
\int_{E} f(x)\,dx = 0 \quad \text{for all measurable } E \text{ with } m(E) = \pi. \quad \text{Must } f = 0 \quad \text{a.e.}\?
\]

6. Let \( f_1, f_2, \ldots \) be a sequence of functions in \( L_2[0,\pi] \) such that

\[
\int_{0}^{1} |f_n(x)|^2\,dx \leq M < \infty \quad \text{for all } n.
\]

Show there is a subsequence \( f_{n_k} \) such that

\[
\lim_{k \to \infty} \int_{0}^{1} f_{n_k}(x)g(x)\,dx \quad \text{exists for all } g \in L_2[0,1].
\]
Complex Analysis

1. Let $f_1, f_2, \ldots, f_n$ be holomorphic in the open unit disc $D$.

   (i) Find the $f_\ell$ if
   
   $$|f_1|^2 + |f_2|^2 + \cdots + |f_n|^2 = 1 \quad \text{in} \quad D.$$ 

   (ii) Find the $f_\ell$ if
   
   $$|f_1|^2 + |f_2|^2 + \cdots + |f_n|^2 = |z|^2 \quad \text{in} \quad D.$$ 

2. Let $f$ be holomorphic in $|z| < 1 + \delta$. Suppose $|f(1)| \geq |f(z)|$ for $|z| \leq 1$.
   Show $f'(1) \neq 0$ unless $f$ is constant.

3. Let $f$ be holomorphic in an annulus $r < |z| < R$. Suppose $|f(z)| = |z|^\lambda$ for all $z$ in the annulus. Show $\lambda$ is an integer.

4. Let $u$ be a $C^3$ real valued function defined for $x^2 + y^2 < 1$. Suppose that $\Delta u(0,0) > 0$. Show there is a harmonic function $v$ such that $v(0,0) = u(0,0)$ and $v(x,y) \leq u(x,y)$ in some neighborhood of $(0,0)$.
   Hint: Look at the Taylor expansion of $u$. 
5. Find a conformal map of the complement (in the sphere) of the line segment \([-1,1]\) onto the strip \(\text{Im } z > 0, \quad -\frac{\pi}{2} < \text{Re } z < \frac{\pi}{2}\).

6. Find \(\int_{|z|=1} \frac{dz}{\sqrt{2z^2+2z+1}}.\)
Complex Analysis

1. Show that for every neighborhood $N$ of an isolated essential singularity for $f$, $f(N)$ is dense in $\mathbb{C}$.

2. Let $f(z)$ be non-constant and holomorphic in the closed annulus $A < |z| < B$, with $|f(z)| = 1$ if $|z| = A$ or $B$. Show that $f$ vanishes at least twice (counting multiplicities) in $A < |z| < B$.

3. Evaluate $\int_0^\infty \frac{dx}{x^2+3x+2}$ by considering a contour integral of $\frac{\log z}{z^2+3z+2}$.

4. Let $f(z) = \int_0^1 \frac{\log(1+tz)}{\sqrt{1+tz}} |\sin 100t|dt$.

Show $f$ is holomorphic for $\text{Re } z > -1$. 
5. For each of the following determine if there is a function $f(z)$, holomorphic in the open unit disc, which satisfies the given property.

(a) $f\left(\frac{1}{n^2}\right) = \frac{1}{n^3}$ \hspace{1cm} n = 2, 3, 4, ....

(b) $f(1 - \frac{1}{n^2}) = \frac{1}{n^3}$ \hspace{1cm} n = 2, 3, 4, ....

(c) $f$ is bounded, $f(0) = 1$ and $f(1 - \frac{1}{n^2}) = 0$ \hspace{1cm} n = 2, 3, 4, ....

(d) $|f(z)| \leq 1$ for all $z$, $f\left(\frac{1}{3}\right) = 0$, $f(0) = \frac{1}{2}$.

6. Is there a sequence of polynomials $P_n(z)$ such that

$$\lim_{n \to \infty} P_n(z) = \begin{cases} 
2 & \text{if } z = 0 \\
1 & \text{if } z \neq 0
\end{cases}$$
Real Analysis

1. Let the Borel set $E$ satisfy $0 < m(E \cap I) < m(I)$ for every nonempty, bounded open interval $I \subset \mathbb{R}$. Must $E$ be of infinite measure?

2. Let $f \in L_1(\mathbb{R})$. Show that for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\left| \int_E f(x) \, dx \right| < \varepsilon$ whenever $m(E) < \delta$.

3. Fatou's Lemma states that under certain conditions, the $\liminf$ of the integrals of a sequence of functions is at least as large as the integral of the $\liminf$. State the condition and prove Fatou's Lemma in this form. Give a counter-example where the condition is absent.
4. Let $f_1, f_2, \ldots$ be measurable functions in the open interval $0 < x < 1$. Suppose

(i) $0 < f_n < \infty$

(ii) $\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0$.

Must there be a point $\xi$, $0 < \xi < 1$, such that

$$\limsup_{n \to \infty} f_n(\xi) < \infty ?$$

5. Let $\mu$ be a positive measure, and $f_1, f_2, \ldots$ functions in $L_1(\mu)$. Which is larger,

$$\left( \sum_{\ell=1}^{\infty} |f_\ell| \, d\mu \right)^{1/2} \quad \text{or} \quad \int \left( \sum_{\ell=1}^{\infty} |f_\ell|^2 \right)^{1/2} \, d\mu ?$$

6. Prove that $L_\infty(0,1)$ is complete.
Real Analysis

I. Let $g$ be a locally bounded function defined on $\mathbb{R}$ such that:
$f(x+y) = f(x) + f(y) + xy$ for all $x$ and $y \in \mathbb{R}$, and such that $f(1) = 1$. Determine what $f$ must be.

(A function $g$ is said to be locally bounded if for every $x \in \mathbb{R}$ there exists a neighborhood of $x$ on which $f$ is bounded.)

II. Let $S$ be a bounded set in $\mathbb{R}$ and let $G$ be a collection of intervals of $\mathbb{R}$ such that for every $x \in S$ and every $\delta > 0$, there exists an interval $I \in G$ with length less than $\delta$ and $x \in I$. Show that there is a subcollection $G_0$ of $G$ which is pairwise disjoint and such that almost every point of $S$ is included in one of the intervals in $G_0$.

Is it possible to cover every (and not only almost every) point of $S$?

III. Compute the maximum of $\left| \int_0^\pi f(x) \sin x \, dx \right|$ where $f$ ranges over all measurable functions on $(0, \pi)$ with:
$\int_0^\pi |f|^2 \leq 1$, $\int_0^\pi f(x) \, dx = 0$ and $\int_0^\pi xf(x) \, dx = 0$. 

$a$
IV. For every function \( f \) defined (possibly only almost everywhere) on (0 1], we define \( \mathcal{F} \) to be the function defined (a.e.) on \( \mathbb{R} \), with period 1, which coincides with \( f \) on (0 1].

1) Let \( f \in L^1(0 1) \). For \( k \in \mathbb{Z} \) set \( e_k(x) = e^{2\pi ikx} \). Let \( f * e_k \) be defined on (0 1) by:

\[
f * e_k(x) = \int_0^1 \mathcal{F}(x-y)e_k(y)dy.
\]

(This is the convolution of periodic functions.) Show that \( f * e_k = ce_k \) for some constant \( c \).

2) Let \( M \) be a closed subspace of \( L^1(0 1) \), \( M \neq \{0\} \). For \( f \in M \) and \( y \in \mathbb{R} \), let \( f_y \) be the function defined on (0 1) by \( f_y(x) = \mathcal{F}(x+y) \). Suppose that for all \( f \in M \) and \( y \in \mathbb{R} \), \( f_y \in M \). Show that there is an integer \( k \) such that \( e_k \in M \).

V. Let \( f(x,y) \) be a \( C^{\infty} \) function defined on \( \mathbb{R}^2 \), 1 periodic in \( x \) and \( y \) (i.e. \( f(x+k,y+\ell) = f(x,y) \) for every \( (k,\ell) \in \mathbb{Z}^2 \)). Find a necessary and sufficient condition in order that there exist \( C^{\infty} \) functions, \( g \) and \( h \), 1 periodic in \( x \) and \( y \) so that

\[
f = \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y}.
\]

VI. Let \( f \) be defined on \( Q = [0 1] \times [0 1] \) to be \( f \left( \frac{p_1}{q_1}, \frac{p_2}{q_2} \right) = \frac{1}{q_1 + q_2} \) at rational lattice points \( (p_1/q_1, p_2/q_2 \) in lowest terms), \( f(x,y) = 0 \) elsewhere. Show that \( f \) is Riemann integrable on \( Q \).
Complex Analysis

I. Give an example of a holomorphic function $f$, defined in a neighborhood of 0 in $\mathbb{C}$, which is real along the curve $\mathcal{P}$ defined by $y = x^2$. [Hint: It may be easier to get $f$ as the inverse map to some mapping.]

Is it possible that a nonconstant holomorphic function $g$, defined in a neighborhood of 0, be real on $\mathcal{P}$ and on the real axis?

II. Let $\Delta$ be the open disk of radius 1, with center at the point 1, in $\mathbb{C}$.

(1) Let $u$ be a positive harmonic function on $\Delta$. Why is $\int\int_{\Delta} u \, dx \, dy < +\infty$?

Give an example to show that, however, $u$ need not be bounded.

(2) Let $\Omega = \{z \in \mathbb{C}, z^2 \in \Delta, \text{Re } z > 0\}$. Sketch $\Omega$, showing very precisely the shape of $\Omega$ in a neighborhood of 0.

Let $u$ be a positive harmonic function on $\Omega$. What can you say (similar to what has been shown in 1)?

III. (1) Let $P$ be a polynomial of degree $n$. Show that for all $z \in \mathbb{C}$ such that $|z| \geq 1$ one has:

$$|P(z)| \leq \left[ \sup_{|\zeta| = 1} |P(\zeta)| \right] |z|^n.$$

(2) Under which condition is it true that, for $|z| \geq 1$, we have also:

$$\left[ \inf_{|\zeta| = 1} |P(\zeta)| \right] |z|^n \leq |P(z)|.$$
IV. (1) Let \( u \) be a harmonic function defined on \( \mathbb{R}^2 \), and \( p \in [1, \infty) \) such that
\[
\iint_{\mathbb{R}^2} |u(x,y)|^p dx \, dy < +\infty.
\]
Show that \( u = 0 \).

(2) Let \( \Omega \) be an open set in \( \mathbb{C} \) and \( u \) be a harmonic function defined on \( \Omega \), let \( p \in [1, \infty) \). If
\[
\iint_{\Omega} |u(x,y)|^p dx \, dy < +\infty,
\]
estimate \( u(x,y) \) in terms of
\[
\iint_{\Omega} |u|^p \quad \text{and} \quad d \quad \text{the distance of} \quad (x,y) \quad \text{to the boundary of} \quad \Omega.
\]

(3) In (2) take for \( \Omega \) the upper half plane defined by \( y > 0 \). Show that if
\[
\iint_{\Omega} |u|^p < +\infty
\]
them \( u(x,y) \) tends to 0 as \( y \) tends to \( \infty \).

V. Let \( \mathcal{O} \) be a connected open set in \( \mathbb{C} \). Let \( f \) be a holomorphic function defined on \( \mathcal{O} \). Show that the following are equivalent:

(i) There exists \( g \) holomorphic on \( \mathcal{O} \) so that \( f = e^g \).

(ii) For every integer \( n, n \neq 0 \), there exists \( g_n \) holomorphic on \( \mathcal{O} \) such that
\[
f = g_n^n.
\]

VI. Let \( U \) be the open unit disk in \( \mathbb{C} \). Let \( f \) be a holomorphic function defined on \( U \). Assume that \( f \) is integrable on \( U \). Show that for all \( z \in U \):
\[
f(z) = \frac{1}{\pi} \iint_{U} \frac{f(w)}{(1-z \bar{w})^2} \, dx \, dy(w).
\]

[If you want, consider first the case when \( f \) is holomorphic on a neighborhood of \( \bar{U} \).]
REAL ANALYSIS

1. Find a necessary and sufficient condition on a measure space \((S, \Sigma, \mu)\) in order that \(L^1(S, \Sigma, \mu) \subseteq L^2(S, \Sigma, \mu)\). Prove your assertion.

2. Prove Minkowski's inequality for \(1 < p < \infty\) : \(\| f + g \|_p \leq \| f \|_p + \| g \|_p\).

3. Show: if every set of positive measure in \((S, \Sigma, \mu)\) can be divided into two sets of positive measure, then for every \(A \in \Sigma\), there is a \(B \in \Sigma\), so that

\[
\frac{1}{3} \mu(A) \leq \mu(A \cap B) \leq \frac{2}{3} \mu(A).
\]

4. In \(L^2(0,1)\) one says that a sequence \(\{f_n\}\) converges weakly to \(f\), if for every \(\varphi \in L^2(0,1)\) we have \(\lim_{n \to \infty} \int_0^1 f_n(x) \varphi(x) dx = \int_0^1 f(x) \varphi(x) dx\).

(a) Determine the weak limits of the following sequences

(i) \(f_n(x) = \sin(nx)\)

(ii) \(f_n(x) = \sin^2(nx)\)

(b) Assume that \(\{f_n\}\) is a sequence of real valued functions in \(L^2(0,1)\) such that \(f_n^2 \in L^2(0,1)\) for all \(n\). Assume that \(f_n\) converges weakly to \(f\), and \(f_n^2\) converges weakly to \(f^2\). Show that \(\int_0^1 (f_n - f)^2 dx \to 0\) as \(n \to \infty\).

Is the similar result true in \(L^2(0, \infty)\)?

5. Suppose \(f_n \in L^1(0,1), f_n \geq 0, \int_0^1 f_n(x) dx \leq 1\) and \(\lim_{n \to \infty} f_n(x) = 0\) a.e.

(a) Is it possible that \(\lim_{n \to \infty} \int_0^1 f_n(x) g(x) dx = \int_0^1 g(x) dx\) for all \(g \in L^\infty(0,1)\)?

(b) Is it possible that \(\lim_{n \to \infty} \int_0^1 f_n(x) g(x) dx = \int_0^1 g(x) dx\) for both \(g = \chi_{[0, \frac{1}{2}]}\) and \(g = \chi_{[\frac{1}{2}, 1]}\)?

(c) Is it possible that \(\lim_{n \to \infty} \int_0^1 f_n(x) g(x) dx = \int_0^1 g(x) dx\) for all \(g \in C[0,1]\)?

6. Describe the set of \((p, q, r) \in \mathbb{R}^3\) such that \(p, q, r > 1\) and whenever \(f \in L^p[0, \infty]\), \(g \in L^q[0, \infty]\) and \(h \in L^r[0, \infty]\) then \(fgh \in L^1[0, \infty]\).
August 28, 1990

Do three from real and three from complex.

COMPLEX ANALYSIS

1. Show that there do not exist three real valued functions $u, v, w$, harmonic on the whole plane, such that
   
   (a) $u(0) = v(0) = u(1) = 0$, $v(1) = -1$, and
   
   (b) $w(z) \leq u(z)$, $w(z) \leq v(z)$ for all $z$ in the plane.

2. Let $H(z) = \int_0^\infty \sin(xz)e^{-x^2}dx$.
   
   (a) Show that $H$ is an entire function.
   
   (b) Calculate $H'(0)$.

3. Let $f$ be holomorphic in $0 < |z| < 1$ except for poles $a_n$, with $a_n \to 0$ as $n \to \infty$.
   Show that for any $\omega \in \mathbb{C}$ there is a sequence $z_n \to 0$ such that $f(z_n) \to \omega$ as $n \to \infty$.

4. Prove that $\int_{-\infty}^{+\infty} e^{-(t+it)^2} dt = \sqrt{\pi}$, for all real $x$.

5. (a) Give an example of a non-constant entire function $f$ such that $f(z)$ is real where $z$ is real and real when $\text{Im } z = 1$.
   
   (b) What is the inverse image of the parabola $y = x^2$ under the mapping $\varphi(z) = z + iz^2$?
   
   (c) Find a non-constant entire function that is real on the parabola $y = x^2$.

6. (a) Suppose that $K \subseteq \mathbb{C}$ is compact and has a connected complement and that $z_0 \not\in K$. Show that there is a (holomorphic) polynomial $p(z)$ so that $p(z_0) = 1$ and $|p(z)| \leq \frac{1}{2}$ for all $z \in K$.
   
   (b) Does there exist a function $F$, continuous on $|z| \leq 1$, and holomorphic for $|z| < 1$ such that the image of the unit circle, $|z| = 1$, under $f$ is precisely this “figure eight”?, i.e. the union of the boundaries of two discs that touch at one point.
REAL ANALYSIS
Throughout, \( \mathbb{R} = (\neg \infty, \infty) \) and \( m \) is Lebesgue measure.

1. Let \( E \) be Lebesgue measurable and \( B \subseteq \mathbb{R} \).
   
   (a) Suppose \( A \cap B \) is measurable for all closed \( A \subseteq E \). Is \( E \cap B \) measurable?
   
   (b) Suppose \( A \cup B \) is measurable for all closed \( A \supseteq E \). Is \( E \cup B \) measurable?

2. Let \( r_n : n = 1, 2, 3, \ldots \) be an enumeration of the rationals in \((0, 1)\) and define

\[
f(t) = \sum_{n : t \geq r_n} \frac{1}{2^n (1 - r_n)}.
\]

   (a) Determine the \( t \) in \((0, 1)\) where \( f \) is continuous.
   
   (b) Let \( F(x) = \int_0^x f(t) \, dt \). Where is \( F \) differentiable?

   (c) Evaluate \( F(1) \).

3. Suppose that \( f \) is a bounded measurable function on \( \mathbb{R} \) and \( f(x + 1) = f(x) \) for all \( x \).
   
   (a) Show that for all intervals \([a, b] : \)

\[
\lim_{n \to \infty} \int_a^b f(nx) \, dx = (b - a) \int_0^1 f(x) \, dx.
\]

   (b) Suppose there is an interval \([\alpha, \beta]\) with \( \alpha < \beta \) and an increasing sequence of integers \( n_k \) such that \( \lim_{k \to \infty} f(n_k x) = g(x) \) for almost all \( x \in [\alpha, \beta] \).
   
   Show that \( g \) is constant a.e. on \([\alpha, \beta]\).

   (c) Under the hypotheses of (b) show that \( f \) is constant a.e. on \( \mathbb{R} \).
4. Let $\phi$ be a continuous function with compact support on $\mathbb{R}$ such that 
\[ \int_{-\infty}^{\infty} \phi(t) \, dt = 1. \] Let $f$ be locally integrable on $\mathbb{R}$, and for $y > 0$ put 
\[ f_y(x) = \int_{-\infty}^{\infty} f(x - y t) \phi(t) \, dt. \]

(a) Prove that if $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then 
\[ \lim_{y \to 0} \| f_y - f \|_p = 0. \]
(b) Is the same statement true if $p = \infty$?

5. Let $f$ be a measurable function on the interval $[0, 1]$. For $t > 0$ put 
\[ \lambda_f(t) = m\{x \in [0, 1] : |f(x)| > t\}. \]

(a) Show that for $0 < p < \infty$, 
\[ \int_0^1 |f(x)|^p \, dx = p \int_0^\infty t^{p-1} \lambda_f(t) \, dt. \]

(b) For a given function $f$, show that if there exists a constant $C$ so that for all $t$,
\[ \lambda_f(t) \leq \frac{C}{t}, \]
then 
\[ f \in \bigcap_{p < 1} L^p([0, 1]). \]

6. Let $f$ and $g$ be real valued Lebesgue measurable functions on $\mathbb{R}$ and define 
\[ D(a, b) = m\{x : f(x) < a, g(x) < b\} \] for $(a, b) \in \mathbb{R}^2$. Show how $D$ determines 
\[ m\{x : (f(x), g(x)) \in B\} \] for every Borel set $B \subseteq \mathbb{R}^2$. 
1. Evaluate \[ \int_0^\infty \frac{[\log x]^2}{1 + x^2} \, dx. \]

2. In each of the following give an example of a function \( f(z) \), holomorphic in the open unit disc \( D = \{ z \in \mathbb{C} : |z| < 1 \} \), which satisfies the given conditions or else prove that no such function exists:

   (a) \( f(re^{\pi i/n}) \) is real for \( 0 \leq r < 1 \) and \( n = 1, 2, 3, \ldots \) and \( f \) is not constant.

   (b) \( f \) is bounded on \( D \), \( f \) is not constant and
   \[ f \left( \frac{n^2 - 1}{n^2} \exp\left(\frac{n^2 - 1}{n^2} \pi i\right) \right) = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots \]

   (c) \( f(0) = 1, \ f'(0) = 3 \) and \( \text{Re} f(z) \geq 0 \) for \( z \in D \).

3. Let \( U = \{ z = x + iy \in \mathbb{C}: y > 0 \} \) be the upper half plane. Let \( f \) be holomorphic on \( U \) and suppose
   \[ \int \int_U |f(x+iy)|^2 \, dx \, dy < \infty. \]

   (a) Prove that for every \( x \in \mathbb{R}, \ \lim_{y \to 0} |f(x+iy)| = 0. \)

   (b) Prove that for every \( y > 0 \), \( \int_{-\infty}^\infty |f(x+iy)|^2 \, dx < \infty. \)

   (c) Prove that \( \lim_{y \to 0} y \int_{-\infty}^\infty |f(x+iy)|^2 \, dx = 0. \)
4. Let \( f \) and \( g \) be entire functions. Suppose that for all \( z \in \mathbb{C} \),
\[
f(z)g(z + 1) = f(z + 1)g(z) \quad \text{and} \quad f(z)g(z + i) = f(z + i)g(z).
\]
Let \( Q = \{ z = x + iy \in \mathbb{C} : 0 \leq x \leq 1, \ 0 \leq y \leq 1 \} \).

(a) Show that if \( g \) has no zeros on \( Q \) then, for all \( z, w \in \mathbb{C} \),
\[
f(z)g(z + w) = f(z + w)g(z).
\]

(b) If \( f \) and \( g \) have no zeros on the boundary of \( Q \), show that \( f \) and \( g \) have the same number of zeros on the interior of \( Q \), counted with multiplicity.

5. Let \( W \subseteq \mathbb{C} \) be a non empty, bounded open set. For \( z \in \mathbb{C} \) let
\[
F(z) = \frac{1}{\pi} \int \int_W \frac{dA(\xi)}{z - \xi}.
\]

(a) Show that \( F \) is continuous on \( \mathbb{C} \) and holomorphic on \( \mathbb{C} - \overline{W} \).

(b) Show that \( F \) is not identically zero on the unbounded component of \( \mathbb{C} - \overline{W} \).

(c) Show that \( G(z) = F(z) - \overline{z} \) is holomorphic on \( W \).

6. Let \( U = \{ z = x + iy \in \mathbb{C} : x > 0 \} \) be the right half plane. Let \( f \) be holomorphic on \( U \) and suppose
\[
\lim_{z \to 0, z \in U} f(z) = L.
\]

Show that \( \lim_{z \to 0, z \in U} zf'(z) = 0 \) uniformly on every angle \( |\theta| \leq \alpha < \frac{\pi}{2} \).
1) Show that if \( f \in L^1(\mathbb{R}) \)

\[
\int_{-\infty}^{\infty} |f(x + h) - f(x)| \, dx \to 0 \quad \text{as} \quad h \to 0.
\]

(You must do more than just quote a theorem.)

2) Evaluate

\[
\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx.
\]

3) Evaluate

\[
\oint_{|z|=1} e^{1/z^2} \, dz.
\]

4) Let \( E \) denote the space of continuous functions \( f(x) \) on \( \mathbb{R}^2 \) such that

\[
\lim_{|z| \to \infty} f(x) = 0.
\]

On \( E \) define a continuous linear functional \( L \) by the formula

\[
Lf = \int_0^{2\pi} f(\cos \theta, \sin \theta) \, d\theta.
\]

According to the Riesz representation theorem \( L \) corresponds to a measure \( \mu \).

What is

\[
\mu\{ (x_1, x_2) \mid x_2 > 0 \}?
\]

5) How many zeros does \( z^5 + 2z^2 + 1 \) have in the disc \( |z| \leq 2 \)?
6) Say whether each statement is true or false. If the statement is true, give a proof. (This proof must contain more than just the statement of a theorem.) If the statement is false, give a counterexample.

(i) If \( \{f_n\} \) is a sequence of non-negative functions in \( L^1(R) \) and \( f_n(x) \) converges to \( f(x) \) uniformly on the real line,

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx.
\]

(ii) If \( \{f_n\} \) is a sequence of non-negative functions in \( L^1(R) \), and

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x)dx = 0,
\]

\( f_n(x) \to 0 \) for almost every \( x \).

(iii) Let \( \{f_n\} \) be a sequence of \( L^1 \) functions. Suppose \( f_n(x) \to f(x) \) almost everywhere as \( n \to \infty \), and

\[
\int_{0}^{1} |f_n(x)|dx \to \int_{0}^{1} |f(x)|dx
\]

as \( n \to \infty \). Then

\[
\int_{0}^{1} |f(x) - f_n(x)|dx \to 0
\]

as \( n \to \infty \).

7) Suppose \( \phi \) is a \( C^\infty \) function with compact support on \( R^1 \) and that \( f \) is continuous on \( R^1 \). Assume also that

\[
\int_{-\infty}^{\infty} \phi(x)dx = 1.
\]

Show

\[
\lim_{\epsilon \to 0} \frac{1}{\kappa} \int_{-\infty}^{\infty} f(x-y)\phi\left(\frac{y}{\epsilon}\right)dy = f(x)
\]

for every \( x \).
8) Let $C$ be the complex plane. Find a conformal mapping of the open first quadrant of $C$ onto the open unit disc.

9) Let $f(x)$ be Lebesgue measurable on $R^1$. Define

$$g(x, y) = f(x - y)$$

on $R^2$. Show that $g$ is Lebesgue measurable on $R^2$.

10) (i) Show that if $u$ is a bounded harmonic function in $R^2$ then $u$ is constant.

Suppose $u$ is a $C^2$ function on $R^2$. A consequence of Green's Theorem is the formula

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})d\theta - \frac{1}{2\pi} \int \int_{|z| \leq 1} \Delta u(z) \log \frac{1}{|z|} dx dy. \tag{*}$$

(Here $z = x + iy$, with the standard identification of $R^2$ and $C$.) You may use (*) in the remaining parts of this problem.

(ii) Let $v$ be a $C^2$ function defined on $R^2$. Assume $v$ is subharmonic (i.e. $\Delta v \geq 0$) and that $|v(z)| \leq 1$ for $|z| \leq 1$. For $\epsilon$ in $(0, 1)$ show that

$$\int \int_{|z| \leq \epsilon} \Delta v(z) dx dy \leq \frac{(2\pi + 1)}{\log \frac{1}{\epsilon}}.$$

(iii) For $\lambda > 0$, let $u(z)$ be a $C^2$ function on $R^2$. For $\lambda > 0$ set $v_\lambda(z) = u\left(\frac{z}{\lambda}\right)$.

Find the relationship between

$$\int \int_{|z| \leq \lambda R} \Delta v_\lambda(z) dx dy$$

and

$$\int \int_{|z| \leq R} \Delta u(z) dx dy.$$

(iv) Let $u$ be a $C^2$ subharmonic function on $R^2$. Show that if $u$ is bounded on $R^2$, then $u$ is constant.
1. Suppose that \( f \in C^1(0, \infty) \) and
   a) \( \int_0^\infty t |f'(t)|^2 dt < \infty \)
   b) \( \lim_{x \to -\infty} \frac{1}{x} \int_0^x f(t) dt = L \)

Show that \( \lim_{x \to -\infty} f(x) = L \).

Hint: First show that for \( 0 < s < t < \infty \)
\[
\int_s^t f(x) dx = tf(t) - sf(s) - \int_s^t rf'(r) dr.
\]

2. Suppose that \( f \in L^p(0, \infty), 1 < p < \infty \).
   a) Show that \( \int_0^\infty f(t) dt \leq \|f\|_{p} |x|^{1 - \frac{1}{p}} \).
   b) Show that \( \lim_{x \to -\infty} \frac{1}{x^{1 - \frac{1}{p}}} \int_0^x f(t) dt = 0 \).

3. Suppose \( f \in L^p(0, \infty), 1 < p < \infty \), define \( (Tf)(x) = \int_0^1 f(tx) dt \).

Show that there is a constant \( C_p \) so that \( \|Tf\|_p \leq C_p \|f\|_p \).

4. Suppose that \( \varphi \) is a continuous increasing function of \( x \), show that if \( a < b \)
   then \( \int_a^b \varphi'(x) dx \leq \varphi(b) - \varphi(a) \).

5. Suppose \( 1 < p < \infty \), \( f_n \in L^p(-\infty, \infty) \) and \( \|f_n\|_p \leq C < \infty \) and \( f_n \to f \) a.e.

Show that \( \int_{-\infty}^\infty f_n g dx \to \int_{-\infty}^\infty f g dx \) for all \( g \in L^q(-\infty, \infty)^{1/p} + \frac{1}{q} = 1 \).

6. Evaluate \( \sum_{n=1}^\infty \frac{1}{n^2 + 1} \).

7. For \( 0 < \alpha < 1 \), evaluate
\[
\int_0^\infty \frac{x^\alpha}{1 + x^2} dx.
\]

8. Suppose \( F \) is holomorphic in \( |z| < 1 \) and \( |F(z)| \leq 1 \) for all \( |z| < 1 \). Show that \( |F'(z)| \leq \frac{1}{1 - |z|} \).

9. Show that there exist a sequence of holomorphic polynomials \( p_n(z) \) in the complex variable \( z \) such that \( p_n(z) \to 1 \) uniformly for all \( z \) in the unit interval \( [0, 1] \) on the real axis and \( p_n(z) \to 0 \) for all other \( z \in \mathbb{C} \).

10. Let \( \gamma > 0 \) and for \( |z| < 1 \) set \( F(z) = \sum_{n=1}^\infty \frac{z^n}{n^\gamma} \). Show that \( F \) has an analytic continuation into the complex plane minus the real axis cut from \( 1 \) to \( +\infty \).

(Hint: Show that \( \frac{1}{n^\gamma} = C_\gamma \int_0^\infty e^{-nt}t^{\gamma-1} dt \), for some constant \( C_\gamma \neq 0 \).
1. Let $\mu$ be a positive Borel measure with compact support in $\mathbb{R}^n$ and $0 < \alpha < n$. Show that $\int \frac{d\mu(y)}{|x-y|^\alpha} < \infty$ for almost all $x$ in $\mathbb{R}^n$, with respect to Lebesgue measure.

2. Let $f \in L^1(\mathbb{R})$ and assume that $\lim_{h \to 0} \frac{1}{h} \int_{-\infty}^{\infty} |f(x+h) - f(x)|dx = 0$. Show that $f \equiv 0$ a.e.

3. Suppose $f_n$ is a sequence of measurable functions defined on $\mathbb{R}$ such that $f_n(x) \to 0$ as $n \to \infty$ for almost all $x$. Suppose further that $\exists g \in L^1(\mathbb{R})$ such that $|f_n(x)| \leq g(x)$ for all $n$ and $x$. Show that for any $\epsilon > 0 \exists$ a measurable set $E_\epsilon$ such that $|E_\epsilon| < \epsilon$ and $f_n \to 0$ uniformly on $\mathbb{R} \setminus E_\epsilon$.

4. For $f \in L^1(0, \infty)$ define

$$Tf(x) = \frac{1}{x} \int_0^x f(t) dt.$$ 

Show that $\exists C > 0 \exists$

$$\int_0^{\infty} |Tf(x)|^2 dx \leq C \int_0^{\infty} |f(x)|^2 dx$$

for all $f \in L^2(0, \infty)$.

5. Suppose $E \subseteq \mathbb{R}$ is measurable, $|E| > 0$

a) Show that $\lim_{h \to 0} \frac{|E \cap (x,x+h)|}{h} = \chi_E(x)$ for almost all $x$.

b) Show that $\exists E_0 \subseteq E$, $|E_0| > 0$ and $N_0 > 0$ such that

$$n|E \cap (x,x + \frac{1}{n})| \geq 1/2$$

for all $x \in E_0$ and all $n \geq N_0$.

6. Show that all of the zeros of

$$p(z) = 3z^3 - 2z^2 + 2iz - 8$$

lie in the annulus

$$1 < |z| < 2.$$
7. Suppose that \( f(z) \) is entire and not constant. Show that the range of \( f \) is dense in the plane.

8. Suppose \( f \) is holomorphic in the half plane \( \text{Im} z > 0 \) and that \( \lim_{z \to 0} f(z) = L \) exists. Show that for each \( \epsilon > 0 \)
   \[
   \lim_{\epsilon \to 0} \text{Re} f'(z) = 0.
   \]

9. Suppose \( f(z) = u(z) + iv(z) \) is an entire function, \( v(0) = 0 \).
   a) Show that if \( |z| < R \) then
   \[
   f(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{Re^{i\theta} + z}{Re^{i\theta} - z} u(Re^{i\theta})d\theta.
   \]
   b) Suppose \( \exists A, B, \alpha > 0 \)
   \[
   |u(z)| \leq A + B|z|^{\alpha}.
   \]

   Show \( \exists C, D > 0 \)
   \[
   |f(z)| \leq C + D|z|^{\alpha}
   \]

10. a) Show that \( \sqrt[4]{z^4 - 1} \) can be defined to be a single valued holomorphic function for \( |z| > 2 \).
   b) Calculate
   \[
   \int_{|z|=3} \frac{dz}{\sqrt[4]{z^4 - 1}}
   \]
1. Let \( f_n \) be a sequence of functions in \( L^1(\mathbb{R}) \) that converges almost everywhere on \( \mathbb{R} \) to a function \( f \in L^1(\mathbb{R}) \) as \( n \to \infty \). Assume also that \( \lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)|dx = \int_{\mathbb{R}} |f(x)|dx \). Prove that
\[
\lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x) - f(x)|dx = 0.
\]

2. Let \( f_n \geq 0 \) and \( f \geq 0 \) be functions in \( L^1(\mathbb{R}) \) such that \( f_n \to f \) almost everywhere, and \( \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx \). Prove that for every measurable set \( E \subset \mathbb{R} \) we have
\[
\lim_{n \to \infty} \int_E f_n(x)dx = \int_E f(x)dx.
\]

3. Set \( p(x) = x^2/(1 + x^2) \).
(a) Show that for all \( \delta > 0 \) sufficiently small, the function \( x \to f(x + \delta p(x)) \) belongs to \( L^1(\mathbb{R}) \) for every \( f \in L^1(\mathbb{R}) \).
(b) Show that for all \( f \in L^1(\mathbb{R}) \)
\[
\lim_{\delta \to 0} \int_{\mathbb{R}} |f(x + \delta p(x)) - f(x)|dx = 0.
\]

4. Let \( K \) be a continuous function on \( \mathbb{R} \times \mathbb{R} \) and let \( C < \infty \) be a constant such that \( \int_{\mathbb{R}} |K(x,y)|dy \leq C \) for all \( y \in \mathbb{R} \), and \( \int_{\mathbb{R}} |K(x,y)|dy \leq C \) for all \( x \in \mathbb{R} \). For \( f \in L^\infty(\mathbb{R}) \) we define
\[
(Tf)(x) = \int_{\mathbb{R}} K(x,y)f(y)dy.
\]
Prove that there is a finite constant \( M \), depending only on \( C \), such that for all \( f \in L^\infty(\mathbb{R}) \) we have
\[
||Tf||_L \leq M||f||_L.
\]

5. Let
\[
f_n(x) = \frac{1 + x^{2n}}{(1 + x^2)^n}, \quad x \in \mathbb{R}, \ n = 1, 2, 3, \ldots
\]
(a) Find \( \lim_{n \to \infty} f_n(x) = f(x) \) for \( x \in \mathbb{R} \).
(b) On what intervals is the convergence \( f_n \to f \) uniform?
6. Let $D \subset \mathbb{R}^3$ be a connected bounded domain with smooth boundary, and let $f, g \in C^2(\overline{D})$.

(a) Apply the divergence theorem to the vector field $F = f \nabla g$ to prove the identity
\[
\iint_D \nabla f \cdot \nabla g \, dV + \iint_D f \Delta g \, dV = \iint_{\partial D} f \frac{\partial g}{\partial n} \, dA.
\]

(b) Prove that, if $g \in C^2(\overline{D})$ is harmonic on $D$ and the normal derivative of $g$ on $\partial D$ vanishes identically, then $g$ is a constant.

7. Show that the integral
\[
F(a, b) = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx
\]
converges for every $0 < a < b < \infty$, and find its value.

8. Let $D \subset \mathbb{C}$ be a domain such that $\overline{D} \neq \mathbb{C}$. Suppose that $f$ is a bounded holomorphic function on $D$, and there is a constant $C > 0$ such that
\[
\limsup_{z \to \zeta} |f(z)| \leq C \quad \text{for all} \quad \zeta \in \partial D.
\]
Prove that $|f(z)| \leq C$ for all $z \in D$.
(Hint: consider the function $f(z)^n/(z - a)$ for some $a \notin \overline{D}$.)

9. Evaluate the following integral:
\[
\int_0^\infty \frac{dx}{x^3 + 1}.
\]
(Hint: Apply the method of residues to the function $F(z) = \log z/(z^3 + 1)$.)

10. Construct in an explicit way (by an infinite series) a meromorphic function $f(z)$ on $\mathbb{C}$ that has a simple pole at every point $z_n = in$ $(n \in \mathbb{Z})$, with $\text{Res}_n f = n$, and has no other singularities. Justify the convergence!

11. Let $F$ be an entire analytic function on $\mathbb{C}$ that is periodic, with a complex period $\omega \neq 0$ (i.e., $F(z + \omega) = F(z)$ for all $z$). If $g$ is an analytic function with isolated singularities on a domain $D \subset \mathbb{C}$, prove that all singularities of $F \circ g$ on $D$ are essential.
1. For \( n = 1, 2, \ldots \), let \( f_n(x) \) be functions on \( \mathbb{R} \) defined by
\[
 f_n(x) = \frac{x^n}{1 + x^{2n}}.
\]

i) For what values of \( x \) does \( \sum_{n=1}^{\infty} f_n(x) \) converge?

ii) In what intervals of \( x \) does
\[
 \sum_{n=1}^{\omega} f_n(x)
\]
converge uniformly?

2. Let \( f(x) \) be a Lebesgue measurable function on \( \mathbb{R} \). For \((x, y) \in \mathbb{R}^2\), define
\[
 g(x, y) = f(x - y).
\]
Show \( g(x, y) \) is Lebesgue measurable on \( \mathbb{R}^2 \).

3. Let \( f \) be a positive continuous function on the interval \([a, b] \subset \mathbb{R} \). Show that the sequence
\[
 x_n = \left( \int_a^b f(x)^n \, dx \right)^{1/n}
\]
converges to \( \sup \{ f(x) : a \leq x \leq b \} \).

4. Let \( f \) be a holomorphic function on \( \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) such that \( \Re f(z) \geq 0 \).
What kind of singularity can \( f \) have at the origin? Justify your answer!

5. Using residues calculate the integral
\[
 \int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 \, dx.
\]

6. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function of class \( C^2 \) such that for every harmonic function
\( g : \mathbb{C} \to \mathbb{R} \), the function \( f \circ g \) is also harmonic. What can you say about \( f \)?
7. i) Show that if \( a_n \geq 0 \),
\[
\prod_{n=1}^{N} (1 + a_n)
\]
converges to a non-zero limit if and only if \( \sum_{n=1}^{\infty} a_n \) converges.

ii) Is the statement in part i) still valid if the hypothesis \( a_n \geq 0 \) is dropped? Give either a proof or counterexample.

8. Let \( f(x) \in L^p(R) \) and assume
\[
\|f(x + y) - f(x)\|_{L^p(dx)} \leq C|y|.
\]
Show there are constants \( C_1 \) and \( C_2 \) and functions \( f_\epsilon(x) \) such that \( f_\epsilon(x) \in L^p \), \( f_\epsilon(x) \in C' \) and \( f'_\epsilon(x) \in L^p \) such that
\[
\|f_\epsilon(x) - f(x)\|_{L^p} \leq C_1 \epsilon
\]
and
\[
\|f'_\epsilon(x)\|_{L^p} \leq \frac{C_2}{\epsilon}.
\]
Hint: Let
\[
f_\epsilon(x) = \frac{1}{\epsilon} \int f(x - y) \phi\left(\frac{y}{\epsilon}\right) dy
\]
for an appropriate \( \phi \).

9. Let \( D = R^3 - L \) where \( L \) is the line \((0,0,z), \ z \in R \). Let \( \vec{F} \) be the vector field
\[
\vec{F} = -\frac{y}{x^2 + y^2} i + \frac{x}{x^2 + y^2} j + xz k.
\]
Determine all possible values of the path integral
\[
\int_{(0,2,0)}^{(0,2,0)} \vec{F} \cdot d\vec{R}
\]
along paths that lie in \( D \).
1. Let \( f_n(x) = \sin(nx) \) for \( n \in \mathbb{Z}_+ \), \( x \in \mathbb{R} \). Show that there is no sequence \( \{n_k\} \subset \mathbb{Z}_+ \) such that the sequence \( f_{n_k}(x) \) converges for all \( x \) in a set of positive Lebesgue measure as \( k \to \infty \).

2. For \( n \in \mathbb{Z}_+ \) let

\[
f_n(x) = \int_1^n te^{it^2} dt.
\]

Show that for every \( \epsilon > 0 \) the sequence of functions \( f_n(x) \) converges uniformly in \( x \geq \epsilon \) as \( n \to \infty \).

\( \checkmark \)

3. Let \( f \) be a holomorphic function in the unit disc \( U = \{z \in \mathbb{C} : |z| < 1\} \) such that \( f(0) \neq 0 \). Prove that there is an \( \epsilon > 0 \) such that for every \( c \in \mathbb{C} \) satisfying \( 0 < |c| < \epsilon \) the equation \( z^m = cf(z) \) has precisely \( m \) distinct solutions in \( U \).

\( \checkmark \)

4. Let \( F = (f, g) : U \to \mathbb{R}^2 \) be a smooth mapping, defined in a neighborhood \( U \subset \mathbb{R}^2 \) of the origin \( (0,0) \), such that the image \( F(U) \) has Lebesgue measure zero and \( \partial f / \partial x(0,0) \neq 0 \). Prove that there is a smaller neighborhood \( V \subset U \) of \( (0,0) \) such that \( F(V) \) is contained in a smooth curve \( \gamma \subset \mathbb{R}^2 \).

5. Let

\[
F(x) = \int_0^\infty \frac{1 - e^{-xt^2}}{t^2} dt, \quad x > 0.
\]

(a) Calculate \( F'(x) \).

(b) Use (a) to evaluate \( F(x) \). (Justify all calculations in both parts.)

6. Evaluate the integral

\[
\int_{-\infty}^{+\infty} \frac{e^{px}}{1+e^{ix}} \, dx, \quad 0 < p < 1.
\]

(Hint: Integrate around the rectangle with vertices \( -R, R, R+2\pi i, -R+2\pi i \).)
7. Suppose $f \in L^1(0,1)$ and $f(x) > 0$ for all $0 < x < 1$. Prove that for every $\alpha$, $0 < \alpha < 1$,

$$\inf_{|E|=\alpha} \int_E f(x) dx > 0.$$ 

8. For each invertible $n \times n$ matrix $A$ we set $F(A) = A^{-1}$. Find the formula for the directional derivative $D F(A) \cdot B$ of $F$ at the matrix $A$ in the direction of the matrix $B$. (Hint: Consider first the case when $A$ is the identity matrix.)

9. For each $s \in \mathbb{C}$ with $\Re s > 0$ we define

$$H(s) = s \int_0^\infty t^s e^{-t} \frac{dt}{t}.$$ 

(i) Show that $H(s)$ is holomorphic in $\Re s > 0$.

(ii) Show that $H(s)$ continues to a holomorphic function in $\Re s > -1$.

(iii) Show that $H(s)$ does not continue to a holomorphic function in $\Re s > -2$.

Here, $\Re s$ denotes the real value of $s$. 
1. Suppose that $a > 1$ is a real number, show that $f(z) = z - e^z + a$ has exactly one zero in the left half plane, $\text{Re } z < 0$.

2. Suppose that $f$ is a twice differentiable real valued function defined on $(0, \infty)$ and that

$$M_j = \sup_{0 < x < \infty} |f^{(j)}(x)|, \quad j = 0, 1, 2, \quad (f^{(0)} = f).$$

Show that $M_1^2 \leq 4M_0M_2$.

**Hint:** First show that if $x, \ h > 0$ that there is a $t$ so that

$$f'(x) = \frac{1}{2h} [f(x + 2h) - f(x)] - hf''(t).$$

3. Suppose that $f \in L^p(0, \infty), 1 < p < \infty$. Show that

$$\lim_{x \to \infty} x^{1/p} \int_x^\infty \frac{f(t)}{t} dt = 0.$$

4. Suppose that $\Delta = \{z : |z| < 1\}$, $f_n$ is holomorphic in $\Delta$ and that $f_n \to f$ uniformly on compact subsets of $\Delta$. Suppose that $f$ is one-to-one on $\Delta$. Show that for each $r < 1$ $\exists N \in f_n$ is one-to-one on $\{z : |z| \leq r\}$, for all $n \geq N$.

5. For $s, t \geq 0$ suppose $K(x, t) \geq 0$, and moreover that
   
   (a) $K(\lambda s, \lambda t) = \frac{1}{\lambda} K(x, t), \lambda > 0,$

   (b) $\int_0^\infty t^{-1/p} K(1, t) dt = \gamma < \infty$ for some $1 < p < \infty$.

Define $Tf(s) = \int_0^\infty f(t)K(s, t) dt$. Show that

$$\|Tf\|_{L^p} \leq \gamma \|f\|_{L^p}.$$
6. Let $E \subseteq R$ be a measurable set of finite positive measure.

(a) Show that

$$\lim_{n \to \infty} \frac{n}{2} \left| E \cap \left( x - \frac{1}{n}, x + \frac{1}{n} \right) \right| = 1 \quad \text{a.e. on } E$$

(b) Show that $\exists$ a subset $E_0 \subseteq E$ and an $N > 0 \exists |E_0| > \frac{|E|}{2}$ and

$$\frac{n}{2} \left| E \cap \left( x - \frac{1}{n}, x + \frac{1}{n} \right) \right| \geq \frac{1}{2}$$

for all $n \geq N$ and all $x \in E_0$.

7. Evaluate: $\sum_{n=1}^{\infty} \frac{1}{1 + n^2}$.

8. Let $\Delta = \{ z : |z| < 1 \}$. Suppose $f$ is holomorphic in $\Delta$ and that

$$\lim_{z \to z \in \Delta} f(z) = L.$$ 

Show that

$$\lim_{z \to 1} (1 - z)f'(z) = 0.$$ 

9. Let $r = \sqrt{x^2 + y^2 + z^2}$, $(x, y, z) \in R^3$. Define $\vec{F}(x, y, z) = \frac{1}{r^3} (x\vec{i} + y\vec{j} + z\vec{k})$.

Find all possible values for

$$\int_S \int \vec{F} \cdot \vec{N} dA$$

where $S \subseteq R^3$ is a smooth closed surface, and $\vec{N}$ is the outward unit normal, and $(0, 0, 0) \notin S$. 
1. For any complex $n \times n$ matrix $A$ define

$$f_L(A) = \sum_{k=0}^{L} (-1)^k \frac{A^k}{(2k)!}.$$ 

(i) Let $\| \cdot \|$ be a norm on the space $M_n$ of complex $n \times n$ matrices $A$. Show that $f_L(A)$ converges to a limit $f(A)$, as $L \to \infty$, with respect to the given norm.

(ii) Show that there is $\delta > 0$ such that the equation $f(A) = I + B$ has a solution $A$, provided that $\|B\| < \delta$.

2. Determine all complex-valued functions $f$, which are continuous in $[0,2]$ and satisfy the condition $\int_0^2 f(x)x^ndx = 0$ for $n = 0,1,2,3,\ldots$.

3. Let $a$ be a decreasing $C^1$-function in $[0,\infty)$ such that $\lim_{t \to \infty} a(t) = 0$.

(i) Show that $\lim_{N \to \infty} \int_0^N a(t) \sin(t)dt$ exists for all $x > 0$.

(ii) For $x > 0$ show that $\lim_{N \to \infty} \int_0^N a(t) \sin(t)dt$ converges uniformly for $x \in [\epsilon, \infty)$.

(iii) Show that uniform convergence fails in $(0,\infty)$, for a suitable choice of $a$.

4. Let $f$ be an increasing continuous function in the interval $[a,b]$. Then it is known that the pointwise derivative $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} := f'(x)$ exists for almost all $x \in [a,b]$. Show that

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

5. Let $f$, $g$ be two measurable functions on the interval $(0,\infty)$ and assume that

$$\int_0^\infty |f(x)| \frac{dx}{x} < \infty, \quad \int_0^\infty |g(x)| \frac{dx}{x} < \infty.$$ 

Show that for almost all $x \in (0,\infty)$

$$\int_0^\infty |f(\frac{x}{t})g(t)| \frac{dt}{t}$$

is finite.
6. For the following pairs of domains \((D, \Omega)\) either find a biholomorphic mapping \(F : D \to \Omega\) or show that such a biholomorphic mapping does not exist.
   (i) \(D = \{z = x + iy : x < -4\}, \Omega = \{z = x + iy : x > 0, |y| < x/2\}\),
   (ii) \(D = \{z = 0 < |z| < 1\}, \Omega = \{z = x + iy : -1 < x < -1\}\),
   (iii) \(D = \{z = x + iy : -1 < x < -1\}, \Omega = \{z : |z| < 1\}\).

7. Let \(f\) be an analytic function on \(A = \{z \in \mathbb{C} : 0 < |z| < 1\}\) satisfying
   \[
   \int_A |f(x + iy)|^2 \, dx \, dy < \infty.
   \]
   Show that \(f\) has a removable singularity at \(z = 0\).
   \textit{Hint:} Estimate the coefficients in the Laurent expansion of \(f\).

8. Let \(p > 0\) and let \(f\) be an entire function satisfying
   \[
   |f(z)| \leq \exp(|z|^p), \quad z \in \mathbb{C}.
   \]
   Show that for \(n = 0, 1, 2, \ldots\)
   \[
   |f^{(n)}(0)| \leq \left(\frac{e^p}{n}\right)^{n/p} n!
   \]

9. For \(f \in L^p(\mathbb{R})\) define
   \[
   B_t f(x) = \int_0^\infty te^{-ty} f(x - y) \, dy.
   \]
   (i) Suppose that \(1 \leq p < \infty\). Show that
   \[
   \lim_{t \to \infty} \left(\int_{-\infty}^\infty |B_t f(x) - f(x)|^p \, dx\right)^{1/p} = 0.
   \]
   (ii) Suppose that \(f\) is bounded and is supported in \([-1, 1]\). Is it true that \(B_t f\) converges uniformly to \(f\)?
1. (i) Does \( p_N = \prod_{n=2}^{N} (1 + \frac{(-1)^n}{n}) \) tend to a nonzero limit as \( N \to \infty \)?
(ii) Does \( q_N = \prod_{n=2}^{N} (1 + \frac{(-1)^n}{\sqrt{n}}) \) tend to a nonzero limit as \( N \to \infty \)?
Explain.

2. Let \( \{f_n\} \) be a sequence of measurable functions defined in \([0,1]\). Show that \( f_n \) converges to 0 in measure if and only if
\[
\lim_{n \to \infty} \int_0^1 \frac{|f_n(x)|}{1 + |f_n(x)|} \, dx = 0.
\]

3. Calculate
\[
\int_C -y^3 \, dx + xy^2 \, dy
\]
where \( C \) is the plane curve given by the equation \( 10x^{12} + 22y^8 = 240 \), with the positive orientation.

4. Suppose \( \alpha \geq 0 \), and let \( f \) be a bounded function on the real line with the property that
\[
|f(x + h) - f(x)| \leq A|h|^\alpha
\]
for all \( h \in \mathbb{R} \) and almost all \( x \in \mathbb{R} \).
Show that there is a constant \( C \) and for each \( t > 0 \) a \( C^1 \)-function \( g_t \) such that
\[
\|f - g_t\|_\infty \leq Ct^\alpha
\]
and
\[
\|g'_t\|_\infty \leq Ct^{\alpha - 1}.
\]

*Hint: Use an approximation of the identity.*

5. Let \( f \) be a function in \( C^1(\mathbb{R}) \) with compact support and let \( b > 0 \). Show that the limit
\[
A_b(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R} \setminus [-\epsilon,\epsilon]} \frac{f(x - y)}{y} \, dy
\]
exists for all \( x \in \mathbb{R} \).
How do \( A_b(x) \) and \( A_c(x) \) differ for \( b \neq c \)?

6. Let \( D = \{z = x + iy \in \mathbb{C} : y > |x|^{1/m} \} \) and let \( F \) be analytic and bounded in \( D \). What can you say about the growth of \( |F^{(n)}(iy)| \) as \( y \to 0 \)?
7. For each of the following compact subsets $K_i$ of $\mathbb{C}$ let $C(K_i)$ be the space of complex-valued continuous functions on $K_i$ (the topology on $K_i$ is the relative topology for subsets of $\mathbb{C}$).

In each case we shall define a subclass $\mathcal{A}_i$ of $C(K_i)$. Prove or disprove that $\mathcal{A}_i$ is dense in $C(K_i)$.

(i) $K_1 = [-1,1] \times \{0\}$ and $\mathcal{A}_1$ is the set of all (finite) linear combinations of the form $a_0 \cos x + \sum_{j=1}^{n} a_j x^j$ with complex coefficients $a_j$.

(ii) $K_2 = \{ z \in \mathbb{C} : |z| = 2 \}$ and $\mathcal{A}_2$ is the set of restrictions to $K_2$ of functions which are holomorphic in $\{ 1 < |z| < 4 \}$.

(iii) $K_3 = \{ z \in \mathbb{C} : |z| \leq 2 \}$ and $\mathcal{A}_3$ is the set of all continuous functions $f$ with the property that $\int_{|z|=1} f(z) z^3 dz = 0$.

8. Calculate

$$\int_{\mathcal{C}} \frac{z^3 + 1}{z^4 + 2z^2 + z + 1} \, dz$$

where $\mathcal{C}$ is the circle with center 0 and radius 3, with the positive orientation.

9. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions in a domain $\Omega \subseteq \mathbb{C}$ which converges to a function $f$, as $n \to \infty$, uniformly on every compact subset of $\Omega$. Suppose that each function $f_n$ has at most $m$ zeros in $\Omega$ for some fixed $m$. Prove that either $f(z) = 0$ for all $z \in \Omega$ or else it has at most $m$ zeros in $\Omega$. 


1. Let \( M(n, \mathbb{R}) \) be the vector space of \( n \times n \) matrices with real entries. Denote by \( \| \cdot \| \) a norm on \( M(n, \mathbb{R}) \). For \( A \in M(n, \mathbb{R}) \) let \( \text{tr}(A) \) be the trace of \( A \) (that is the sum over all entries on the diagonal). Show that there are neighborhoods \( U, V \) of the identity matrix \( I \) such that for every \( A \in V \) there is a unique \( B \in U \) with \( n^{-1} \text{tr}(B^3) = A \).

2. Prove or disprove the following statement:
\[
\lim_{\epsilon \to 0} \iint_{x^2 + y^2 \geq \epsilon^2} \frac{f(x, y)}{(x + iy)^2} \, dx \, dy
\]
exists for every function \( f \in C^2(\mathbb{R}^2) \) with compact support.

Hint: For \( 0 < a < b \), what are the values of
\[
\iint_{a^2 < x^2 + y^2 < b^2} \frac{x}{(x + iy)^3} \, dx \, dy \text{ and } \iint_{a^2 < x^2 + y^2 < b^2} \frac{y}{(x + iy)^3} \, dx \, dy?
\]

3. Let \( \{f_n\}_{n=1}^\infty \) be a sequence of real-valued measurable functions defined on \([0, 1]\) such that
(i) \( \sup_n \sup_{x \in [0, 1]} |f_n(x)| \leq 1 \)
(ii) \( \int_0^1 f_n(x) f_m(x) \, dx = 0 \) if \( m \neq n \).

Let \( A \) be a measurable subset of \([0, 1]\) with positive Lebesgue measure and let \( \epsilon > 0 \). Show that there are at most finitely many integers \( n \) with the property that \( f_n(x) > \epsilon \) for all \( x \in A \).

4. For a Borel set \( E \subset \mathbb{R}^n \) let \( T(E) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x + y \in E\} \). Given two finite Borel measures \( \mu, \nu \) with compact support in \( \mathbb{R}^n \), the convolution \( \lambda = \mu * \nu \) is defined as the Borel measure \( \lambda \) such that \( \lambda(E) = \iint_{T(E)} d\mu(x)d\nu(y) \) for Borel sets \( E \).

(i) Find and prove a formula for \( \int f d\lambda \), for any continuous function with compact support.
(ii) Let \( C_1 \) be the line segment in \( \mathbb{R}^3 \) connecting the origin to \((0, 1, 1)\) and let \( C_2 \) be the line segment in \( \mathbb{R}^3 \) connecting the origin to \((1, 0, 2)\). Let \( \mu_1, \mu_2 \) be arclength measures on \( C_1, C_2 \), respectively, and let \( \lambda = \mu_1 * \mu_2 \). Define \( F(x, y, z) = x + y + z \). Compute
\[
\int F \, d\lambda.
\]

5. Let \( \Omega \subset \mathbb{C} \) be a convex domain and let \( f : \Omega \to \mathbb{C} \) be a nonconstant holomorphic function satisfying \( \text{Re}(f'(z)) \geq 0 \) for all \( z \in \Omega \). Prove that \( f \) is injective on \( \Omega \).

6. For \( a \in \mathbb{R} \), using complex integration, evaluate the integral:
\[
\int_{-\infty}^{+\infty} \frac{e^{-iax}}{1 + x + x^2} \, dx
\]
(i.e., evaluate the Fourier transform of the function \((1 + x + x^2)^{-1} \) at the point \( a \).)
7. Assume that \( \{a_n\} \in \ell^2, \{b_n\} \in \ell^2 \).
(a) Show that
\[
\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{a_n b_m}{m} < \infty
\]

*Hint:* Estimate the sum
\[
\sum_{2^n \leq m < 2^{n+1}} \frac{b_m}{m}
\]
by the Cauchy-Schwarz inequality.

(b) Show that
\[
\sum_{n,m=1}^{\infty} \frac{a_n b_m}{n + m} < \infty.
\]

(*)

(c) If instead of \( \{a_n\} \in \ell^2 \) we assume \( \{a_n\} \in \ell^4 \) (that is \( \sum a_n^4 < \infty \)), what is an appropriate hypothesis on \( \{b_n\} \) so that (*) holds?

8. Let \( \Delta \) be the open unit disk in \( \mathbb{C} \), and for \( \rho > 0 \), let \( \Delta_\rho = \{ z \in \mathbb{C}, |z| < \rho \} \). Let \( M > 0 \).

Give an explicit value of \( \rho > 0 \) so that \( f(\Delta) \supset \Delta_\rho \) for all holomorphic functions \( f \) defined on \( \Delta \), satisfying

1. \( f(0) = 0 \)
2. \( f'(0) = 1 \)
3. \( |f(z)| \leq M \) for all \( z \in \Delta \).

*Hint:* First find explicitly \( r \) and \( A \) depending only on \( M \) such that \( |f(z) - z| \leq A|z|^2 \) for \( |z| \leq r \). Note that you are asked for an explicit value of \( \rho \) but not for the largest possible value.

9. Let \( F \) be an integrable function defined on \( \mathbb{R} \) and let \( G \) be defined by
\[
G(x) = \int_0^1 tF(x + t) \, dt.
\]

(a) Show that \( G \) is a continuous function on \( \mathbb{R} \).

(b) Show that if \( F \) is continuous, then \( G \) is differentiable and \( G' \) is continuous.

(c) Show that the converse holds: if \( G' \) is continuous, then \( F \) is equal almost everywhere to a continuous function.
1. Let $f$ be a real valued function defined in the interval $[-2,2]$.

(i) Assume that $f$ is of class $C^2$. Show that there is $C \geq 0$ such that for $|h| \leq 1$

$$\int_{-1}^{1} |f(x+h) + f(x-h) - 2f(x)| \, dx \leq C h^2.$$ 

(ii) Assume that $f$ is a convex function, no longer assumed to be of class $C^2$. Show that the same holds.

**Hint:** Show first that if $x_0 < x_1 < \ldots < x_N$ and $x_i - x_{i-1} = h$ then

$$\sum_{i=1}^{N-1} |f(x_{i+1}) - 2f(x_i) + f(x_{i-1})| \leq C' |h|.$$ 

**NOTE:** You can consider that it is well known that the restriction of $f$ to any closed interval included in $(-2,2)$ is Lipschitzian, and as usual convex means that for any $x$ and $y \in [-2,2]$, $f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$.

(iii) Would the above hold for any continuous function $f$?

2. (i) Do the following indefinite integrals exist

$$\int_0^\infty t^i \sin t \, dt, \quad \int_0^\infty t^{i-1} \sin t \, dt, \quad \int_0^\infty t^{i-1} \cos t \, dt?$$

(ii) For $\eta \in \mathbb{R}$, set $I(\eta) = \int_0^1 t^\eta \cos t \, dt$. Is $I(\eta)$ a continuous function of $\eta$? What is the limit of $I(\eta)$ as $\eta \to \pm \infty$?

3. Let $D$ be a bounded domain in $\mathbb{R}^2$, bounded by a smooth Jordan curve $C$.

(i) Green’s theorem (i.e., Stokes theorem in dimension 2) and the divergence theorem both relate integration along the curve $C$ and integration on the domain $D$. State both theorems.

(ii) To a pair of (smooth) functions $a(x,y)$ and $b(x,y)$ defined on $C$, associate a vector field $\vec{V}(x,y)$ defined on $C$ such that the flux of $\vec{V}$ across $C$ is equal to

$$\int_C a \, dx + b \, dy.$$ 

And show how Green’s theorem follows from the divergence theorem.

(iii) Let $f$ be a holomorphic function defined on a neighborhood of the closure of $D$ (i.e., a continuously differentiable, complex valued function satisfying $\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0$). Deduce from Green’s theorem that $\int_C f(z) \, dz = 0$, $(z = x + iy)$. (Cauchy theorem).
4. (i) Suppose $f_n$, $h_n$, and $h$ are real valued $L^1$ functions defined on the real line, and suppose $f_n \to 0$ a.e. and $h_n \to h$ a.e. as $n \to +\infty$. Suppose also that $|f_n(x)| \leq h_n(x)$, for every $n \in \mathbb{N}$, and $\int h_n(x) dx \to \int h(x) dx$, as $n \to +\infty$. Prove that $\int f_n(x) dx \to 0$.

Hint: One can apply Fatou’s Lemma to $h_n - |f_n|$. But this is not the only possible approach.

(ii) Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Prove that $\int |f_n(x) - f(x)| dx \to 0$ if and only if $\int |f_n(x)| dx \to \int |f(x)| dx$.

NOTE:

a) One implication above (if or only if) is absolutely immediate, make this clear.

b) You may have seen the results of this problem stated as theorems. However you are asked to give proofs (based on basic results such as Fatou’s Lemma, or the standard version of the Lebesgue dominated convergence theorem).

5. Let $1 \leq p < q < \infty$.

Which of the following statements (i)-(vi) are true, and which are false? Justify all the negative answers by a counterexample, but you do not have to justify the positive answers.

(i) $L^p(\mathbb{R}) \subset L^q(\mathbb{R})$. \(\checkmark\)

(ii) $L^q(\mathbb{R}) \subset L^p(\mathbb{R})$. \(\checkmark\)

(iii) $L^p([0,1]) \subset L^q([0,1])$. \(\checkmark\)

(iv) $L^q([0,1]) \subset L^p([0,1])$. \(\checkmark\)

(v) $\ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$. \(\checkmark\)

(vi) $\ell^q(\mathbb{Z}) \subset \ell^p(\mathbb{Z})$. \(\checkmark\)

Justify your answer to the following question:

(vii) For which $s \geq 1$, $L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^s(\mathbb{R})$?

6. Let $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Let $h$ be a continuous function defined on $\mathbb{R}$, and let $\Phi$ be the map from $\mathbb{R}^2$ into itself defined by:

$$\Phi(x, y) = (x + h(x + y), y - h(x + y)).$$

Determine the area of $\Phi(\Omega)$.

Note: Although you can consider first the case when $h$ is continuously differentiable, you are asked to treat (with justifications) the case when $h$ is merely continuous.
7. (i) By using complex integration, evaluate the limit, as $\epsilon \to 0^+$ and $R \to +\infty$, of the integral:

$$\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{e^{-ix}}{x} \, dx.$$ 

(ii) For $a \in \mathbb{R}$, evaluate the limit, as $\epsilon \to 0^+$ and $R \to +\infty$, of the integral:

$$\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{e^{-iax}}{x} \, dx.$$ 

(You can gain time, by deducing it from (i), for $a \neq 0$).

(iii) Set

$$I_\epsilon(a) = \int_{-\epsilon}^{\epsilon} + \int_{\epsilon}^{1} \frac{e^{-iax}}{x} \, dx \quad \text{and} \quad J_{R}(a) = \int_{-R}^{-1} + \int_{1}^{R} \frac{e^{-iax}}{x} \, dx.$$ 

For $a \in [0, 1]$, does $I_\epsilon(a)$ (resp $J_{R}(a)$) have a uniform limit as $\epsilon \to 0^+$ (resp $R \to +\infty$)?

8. Let $\Delta$ be the unit disk in $\mathbb{C}$, and let $f$ be a holomorphic function defined on $\Delta$, assume that $\int_{\Delta} |f|^2 \, dx \, dy \leq 1$.

(i) Show that $|f(z)| \leq \frac{1}{1-|z|^2}$. (Or show a sharper estimate).

(ii) Conversely, if $|f(z)| \leq \frac{1}{1-|z|^2}$, is $\int_{\Delta} |f|^2$ finite?

(iii) What can you say about the coefficients in the series expansion of $f$ ($f = \sum a_j z^j$)? (An estimate of $|a_j|$, or of $\sum |a_j|^2$, or of some similar quantity).

9. It is known that if $g$ is a $C^1$ function defined on $\mathbb{C}$, there exists $u$, a $C^1$ function defined on $\mathbb{C}$, such that $\frac{\partial u}{\partial \bar{z}} = g$. As usual, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$.

(i) Find $u$ satisfying $\frac{\partial u}{\partial \bar{z}} = g$ for

a) $g(z) = |z|^2$.

b) $g(z) = xy \quad (z = x + iy)$.

(ii) If $g$ has compact support, does there always exist $u$ a function with compact support satisfying:

$$\frac{\partial u}{\partial \bar{z}} = g.$$ 

Hint: you can start by setting $g = \frac{\partial v}{\partial \bar{z}}$, with $v$ conveniently chosen.
Problem 1. In \( \mathbb{R}^3 \), for which real \( p > 0 \) is \( (x_1^2 + x_2^2 + x_3^2)^{-p} \) integrable in a neighborhood of 0?

(ii) integrable at infinity?

Hint: Decompose \( \mathbb{R}^3 \) in ‘annuli’

\[ \Gamma_k = \{(x_1, x_2, x_3) \in \mathbb{R} : 2^k \leq (x_1^2 + x_2^4 + x_3^6) \leq 2^{k+1}\} \]

and estimate the volume of each \( \Gamma_k \).

Problem 2. Suppose that \( a_n > 0 \) for each \( n = 1, 2, 3, \ldots \) and \( \sum_{n=1}^{\infty} a_n = +\infty \). Discuss the convergence resp. divergence of the series

\[ \sum_{n=1}^{\infty} \frac{a_n}{1 + n^p a_n} \]

for each real \( p \geq 0 \). (If for some \( p \) the series can either converge or diverge, depending on the choice of \( a_n \), give an example of both.)

Problem 3. Let \( S \subset \mathbb{R}^3 \) be the plane through the origin with the unit normal vector \( \mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3 \), and let \( S \) be oriented by \( \mathbf{n} \). For \( r > 0 \) we denote by \( C_r \) the circle of radius \( r \) in \( S \) centered at the origin. Let \( \mathbf{T} \) be the unit tangent vector field to \( C_r \) in the positive (counterclockwise) direction.

(i) Prove that for each continuous vector field \( \mathbf{F} \) on \( \mathbb{R}^3 \) we have

\[ \lim_{r \to 0} \frac{1}{r} \int_{C_r} \mathbf{F} \cdot \mathbf{T} \, ds = 0. \]

(Here \( ds \) denotes the arc length on \( C_r \).)

(ii) Suppose in addition that \( \mathbf{F} \) is differentiable at the origin. Prove that the following limit exists:

\[ \lim_{r \to 0} \frac{1}{r^2} \int_{C_r} \mathbf{F} \cdot \mathbf{T} \, ds. \]

(iii) Calculate the limit in (ii) for \( \mathbf{n} = (1, 0, 0) \) and

\[ \mathbf{F}(x) = (e^{x_1}, x_2 \sin x_3, x_3 \cos x_3). \]

Problem 4. Let \( 1 < p < +\infty \). Let \( f_j \) be a sequence in \( L^p(\mathbb{R}) \) such that for all \( j = 1, 2, 3, \ldots \),


(i) $\|f_j\|_p = 1$ (the $L_p(\mathbb{R})$ norm), and

(ii) for almost every $x \in \mathbb{R}$ we have $f_j(x) = 0$ or $|f_j(x)| \geq j$.

Show that for any $g \in L^q(\mathbb{R})$, $(\frac{1}{p} + \frac{1}{q} = 1)$, $\int_R f_j g$ tends to 0 as $j$ tends to $+\infty$.

**Problem 5.** Let $f \in L^1([0,1])$. For $k \in \mathbb{N}$, let $f_k$ be the step function defined on $[0,1]$ by:

$$f_k(x) = k \int_{\frac{j}{k}}^{\frac{j+1}{k}} f(t) \, dt \quad \text{for} \quad \frac{j}{k} \leq x < \frac{j+1}{k}.$$ 

Show that $f_k$ tends to $f$ in $L^1$ norm as $k$ tends to $+\infty$.

Hint: Treat first the case when $f$ is continuous, and use approximation.

**Problem 6.** (This problem can be done even if you have not done the problem 5 above.) In the problem 5 above, is the convergence of $f_k$ to $f$ a dominated convergence in the sense of the Lebesgue dominated convergence theorem?

Hint: Try $f(x) = 1/(x \log^2 x)$.

**Problem 7.** Let $f$ be a holomorphic function, defined on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$, such that:

$$f(0) = 0 \quad \text{and} \quad |f'(z)| \geq 1 \quad \text{for every} \quad z \in D.$$ 

Show that $f(D) \supset D$.

Hint: If a line segment $[0, \zeta]$ is included in $f(D)$, show that there exists a $z \in D$ such that $f(z) = \zeta$ and $|z| \leq |\zeta|$.

**Problem 8.** Let $D$ be the unit disc as in problem 7 above. Suppose that $f$ is a nonconstant holomorphic function on $D$ satisfying $\text{Re} f(z) \geq 0$ for all $z \in D$ and $f(0) = 1$. Prove

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$ 

**Problem 9.** Let $0 < a < 1$. Evaluate the integral

$$I(a) = \int_{-\infty}^{+\infty} \frac{e^{ax}}{1 + e^x} \, dx.$$ 

Hint: Compare $I(a)$ with $\int_{-\infty}^{+\infty} \frac{e^{ax+2\pi i}}{1 + e^x} \, dx$ by a contour integration in $\mathbb{C}$.
Problem I.

(a) Find an explicit value of $\epsilon > 0$ so that for every real number $x \in [0, 1]$,

$$|\sqrt{x} - \sqrt{x + \epsilon}| \leq \frac{1}{200}.$$ 

(b) Find an explicit integer $N$ such that there exists a polynomial $P$ of degree at most $N$ such that for every real number $x \in [0, 1]$,

$$|\sqrt{x} - P(x)| \leq \frac{1}{100}.$$ 

(Hint: You could use the Taylor expansion of the function $\sqrt{x + \epsilon}$ in powers of $(x - 1)$.)

Problem II.

Let $y(t)$ be a continuously differentiable solution to the following initial value problem on the interval $[0, T)$:

$$\frac{dy}{dt} = y^2 \left[ 2 + \sin(e^t + y) \right]$$

$$y(0) = 1.$$ 

Show that $T < 1$.

Problem III.

Let $f$ and $g$ be twice continuously differentiable real functions defined on $\mathbb{R}^2$. Assume that they vanish at $(0, 0)$ and that they do not have any other common zero.

Under which conditions on the partial derivatives of $f$ and $g$ at $(0, 0)$ is the improper integral

$$\iint_{|x|^2 + |y|^2 \leq 1} \frac{dz \, dy}{|f|^{|g|}}$$

convergent?

You are asked a complete discussion.
Problem IV.

On the interval \([-1, +1]\) consider the standard Banach spaces \(L^1\) and \(L^2\) with the usual norms
\[
\|f\|_1 = \int_{-1}^{+1} |f(x)| \, dx; \quad \|f\|_2 = \left( \int_{-1}^{+1} |f(x)|^2 \, dx \right)^{\frac{1}{2}}. \quad \|f\|_1 \leq \sqrt{2} \|f\|_2
\]
(a) How are these two norms related? (Give inequalities, state sharp results, and give justifications.)
(b) Let \(\{f_j\}\) be a sequence of functions in \(L^2\). Assume that \(f_j \geq 0\), that \(\|f_j\|_1 = 2\), and that
\[
\|f_j\|_2 = 2^{-j}.
\]
Show that \(f_j\) tends almost everywhere to the constant 1 on the interval \([-1, +1]\). (Hint: Write \(f_j = 1 + h_j\).)
(c) If we drop the hypothesis that \(f_j \geq 0\), what should replace the above conclusion?

Problem V.

Let \(f\) be a locally integrable function defined on \(\mathbb{R}\). Let
\[
L_f = \{ x \in \mathbb{R} \mid f(x) = \lim_{h \to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} f(t) \, dt \}
\]
be the set at which small averages of \(f\) converge to \(f\).
(a) Is the set \(L_f\) always non-empty? Explain what you know about this. Give a non-trivial example of a function \(f\) and a point \(x \in L_f\) at which \(f\) is not continuous.
(b) For \(t > 0\) define a function \(g_t\) on \(\mathbb{R}\) by \(g_t(x) = \frac{1}{t} \left(1 - \frac{|x|}{t}\right)\) if \(|x| < t\), and \(g_t(x) = 0\) if \(|x| \geq t\).
Graph the function \(g_t\). Suppose \(x_0 \in L_f\). Prove that
\[
\lim_{t \to 0^+} f * g_t(x_0) = f(x_0), \quad (\ast \text{ denotes convolution} ).
\]
Hint: For \(a > 0\), let \(\chi_{[-a, +a]}\) denote the characteristic function of the interval \([-a, +a]\).
Show the following identity, explain its meaning:
\[
g_t(x) = \int_0^1 \left( \frac{\chi_{[-ht, +ht]}(x) \cdot 2h}{2h} \right) dh.
\]

Problem VI.

For each \(t > 0\), let \(f_t\) be a measurable function defined on the real line \(\mathbb{R}\). Let
\[
E = \{ x \in \mathbb{R} \mid \lim_{t \to \infty} f_t(x) \text{ exists} \}.
\]
(a) Show by an example that the set \(E\) need not be measurable.
(b) If each of the functions \(f_t\) is continuous, show that \(E\) is a Borel set and hence measurable.
Problem VIIc.

Let \( f \) be a holomorphic function, defined in a neighborhood of the disk

\[ \Delta = \{ \zeta \in \mathbb{C} \mid |\zeta| \leq 4 \} \]

Suppose that \( f \) does not vanish on the circle \( \{ \zeta \in \mathbb{C} \mid |\zeta| = 4 \} \), and satisfies

\[ \int_{|\zeta|=4} \frac{\zeta^k f'(\zeta)}{f(\zeta)} \frac{d\zeta}{f(\zeta)} = 2^{k+2} \pi i \]

for \( k = 0, 1, 2 \). Find all the zeros of \( f \) in \( \Delta \).

Problem VIIIc.

Let \( D^* = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \} \). Let \( f \) be a non constant holomorphic function on \( D^* \). Assume that \( \text{Im} \ f(z) \geq 0 \) if \( \text{Im} \ z \geq 0 \) and that

\[ \text{Im} \ f(z) \leq 0 \text{ if } \text{Im} \ z \leq 0 \]

(a) Show that if \( z \in D^* \) and \( z \) is not real, then \( f(z) \) is not real. Show that if \( z \in (-1, 0) \cup (0, 1) \), then \( f'(z) \neq 0 \).

(b) Show that either \( f \) has a holomorphic extension to the unit disk satisfying \( f'(0) \neq 0 \), or that \( f \) has a meromorphic extension to the unit disk with a simple pole at 0.

Problem IXc.

It is a well known fact that if \( u \) is a continuous function defined on \( \mathbb{R} \), then

\[ u(0) = \lim_{\tau \to 0^+} \frac{1}{\sqrt{2\pi\tau}} \int_{-2}^{+2} u(t) e^{-\frac{t^2}{2\tau}} dt. \]

(This is just approximation by convolution with a Gaussian kernel, and the limits of integration \( \pm 2 \) could be replaced by others).

Let \( f \) be a holomorphic function on \( \mathbb{C} \). Show that:

\[ f(i) = \lim_{\tau \to 0^+} \frac{1}{\sqrt{2\pi\tau}} \int_{-2}^{+2} f(t) e^{-\frac{(t-i)^2}{2\tau}} dt. \]

(Hint: Change contour of integration. Take the piecewise linear path from \(-2\) to \(-2 + i\), to \( 2 + i \), to \( 2 \)).
Problem VIIr.

Let \( \{f_n\} \) be a sequence of continuous functions defined on the unit interval \([0, 1]\). Suppose that \( \lim_{n \to \infty} \int_0^1 |f_n(t)| \, dt = +\infty \). For each of the following two statements, either give an example of a sequence \( \{f_n\} \) which satisfies the statement, or else prove that the statement is false.

(a) For every continuously differentiable function \( g \) on \([0, 1]\), \( \lim_{n \to \infty} \int_0^1 f_n(t) g(t) \, dt = 0 \).

(b) For every continuous function \( h \) on \([0, 1]\), \( \lim_{n \to \infty} \int_0^1 f_n(t) h(t) \, dt = 0 \).

Problem VIIIr.

Let \( h \) be the function defined on \( \mathbb{R} \) by

\[
    h(x) = \begin{cases} 
        0 & \text{if } x \leq 0; \\
        x^2 & \text{if } x \geq 0. 
    \end{cases}
\]

(a) Determine \( h', h'', \) and \( h''' \), the derivatives of \( h \) in the sense of distributions. If they are the distributions defined by some function or measure, indicate which function or which measure.

(b) Let \( \varphi \) be a smooth function with compact support on \( \mathbb{R} \). Give a simple expression for \( \varphi''' * h \) in terms of \( \varphi \). (As usual, \( * \) denotes convolution). Explain how the result is related to the Taylor formula with integral remainder.

Problem IXr.

Let \( Q = (-1, +1) \times (-1, +1) \subset \mathbb{R}^2 \). Let \( C^1(\overline{Q}) \), (respectively \( C^2(\overline{Q}) \)) denote the space of continuously differentiable (respectively twice continuously differentiable) functions on \( \overline{Q} \), the closure of \( Q \). Let \( H^1(Q) \) denote the Sobolev space of functions defined on \( Q \) which belong to \( L^2(Q) \) and whose first derivatives also belong to \( L^2(Q) \). Let \( j_1 \) and \( j_2 \) be the inclusion maps:

\[
    j_1 : C^1(\overline{Q}) \to H^1(Q),
\]
\[
    j_2 : C^2(\overline{Q}) \to H^1(Q).
\]

Are \( j_1 \) and \( j_2 \) compact operators?
Reminder: All answers have to be carefully justified.

1. (i) Determine $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.
(ii) Compute

$$\lim_{N \to \infty} \frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4N-3} + \frac{1}{4N-1} - \frac{1}{2N}.$$  

Hint: One possible approach is to determine the behavior of $\sum_{k=2N+1}^{4N-1} k^{-1}$ as $N \to \infty$; another approach uses the definition of Euler’s constant.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(x) \geq 0$ for all $x \geq 0$. Consider the two statements

\begin{align*}
(1) & \quad \int_0^\infty f(x) \, dx \text{ converges}, \\
(2) & \quad \sum_{n=0}^\infty f(n) \text{ converges}.
\end{align*}

(i) Discuss the truth of the implications (1) $\implies$ (2) and (2) $\implies$ (1).

(ii) Assume $f$ is continuously differentiable and satisfies $|f'(x)| \leq A$ for some constant $A < \infty$; again discuss the truth of the implications (1) $\implies$ (2) and (2) $\implies$ (1).

(iii) Finally, assume $|f'(x)| \leq A|f(x)|$ for some constant $A < \infty$ and once more discuss the truth of the implications (1) $\implies$ (2) and (2) $\implies$ (1).

3. Let $r > 1$, $a > 0$ and $b > 0$. Show that there is a unique differentiable function $f$ defined on $(-\infty, \infty)$ satisfying $f(0) = 0$ and $f'(x) = a - b|f(x)|^r$ for all $x$. Show that $\lim_{x \to \infty} f(x)$ exists and determine this limit.

4. Let $I = [0, 1]$ and suppose that $f \in L^1(I)$. Let $\{f_n\}$ be a sequence of square-integrable functions with $\|f_n\|_{L^2(I)} \leq 1$ and suppose that $\lim_{n \to \infty} \|f - f_n\|_{L^2} = 0$.

(i) Show that $f \in L^2(I)$ and $\|f\|_{L^2} \leq 1$.

(ii) Do the hypotheses imply that $f_n$ converges to $f$ in $L^2(I)$?

5. Let $f$ and $g$ be nonnegative and measurable on the interval $[0, 1]$. Suppose that

\begin{equation}
(*) \quad f(x)g(x) \geq 1 \text{ for all } x \in [0, 1].
\end{equation}

(i) Show that

$$\int_0^1 f(x) \, dx \int_0^1 g(x) \, dx \geq 1.$$  

(ii) For which choices of positive $p$, $q$ does the assumption (*) imply that

$$\left( \int_0^1 |f(x)|^p \, dx \right)^{1/p} \left( \int_0^1 |g(x)|^q \, dx \right)^{1/q} \geq 1?$$
6. (i) Show that for any $f \in L^1([0,1])$ one has

$$\lim_{\lambda \to \infty} \int_0^1 e^{i\lambda x} f(x) \, dx = 0$$

*Hint:* Prove it first for a suitable class of nice functions.

(ii) Let $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of positive integers. Let $A$ be the subset of $[0,1]$ consisting of all $x$ where $\cos 2\pi n_k x$ converges. Prove that $A$ has measure zero.

*Hint:* For every subset $B$ determine $\lim_{k \to \infty} \int_B \cos 2\pi n_k x \, dx$ and $\lim_{k \to \infty} \int_B [\cos 2\pi n_k x]^2 \, dx$.

7. (i) For $x \in \mathbb{R}$ let

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1 + \xi^2)^2} e^{i\xi x} \, d\xi.$$ 

Compute $K(x)$.

(ii) Write the function $K$ as the sum of two functions $K^+$ and $K^-$ (given by integrals) such that $K^+$ has a continuous extension to the upper half plane which is holomorphic and bounded, and similarly $K^-$ has a continuous extension to the lower half plane which is holomorphic and bounded. Is such a decomposition $K = K^+ + K^-$ unique?

8. (i) Let $P$ be a complex polynomial. Assume that all the roots of $P$ have modulus less than 1, show that all the roots of $P'$ have modulus less than 1.

(ii) Let $h$ be a holomorphic function defined on the unit disk. Is there any relation between the number of zeroes of $h$ and the number of zeroes of $h'$ in the unit disk.

In case of negative answers give examples with $h$ or $h'$ zero free.

9. Compute one of the following integrals:

(i)

$$\int_0^\infty x^{-1/3}(1 + x^2)^{-1} \, dx.$$  

or

(ii)

$$\int_0^\infty \frac{\sin(x^2)}{x} \, dx.$$
6. (i) Show that for any $f \in L^1([0, 1])$ one has

$$\lim_{\lambda \to 0} \int_0^1 e^{i\lambda x} f(x) dx = 0$$

**Hint:** Prove it first for a suitable class of nice functions.

(ii) Let $\{n_k\}_{k=0}^\infty$ be an increasing sequence of positive integers. Let $A$ be the subset of $[0, 1]$ consisting of all $x$ where $\cos 2\pi n_k x$ converges. Prove that $A$ has measure zero.

**Hint:** For every subset $B$ determine $\lim_{k \to \infty} \int_B \cos 2\pi n_k x \, dx$ and $\lim_{k \to \infty} \int_B [\cos 2\pi n_k x]^2 \, dx$.

7. (i) Let

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + x^4} e^{-i\xi x} d\xi.$$ 

Show that there is $C > 0$ such that $K(x) \leq C(1 + |x|)^{-2}$ for all $x \in \mathbb{R}$.

**Note:** You are not required to compute this integral although this is possible.

(ii) Let

$$T_n f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + x^4} \hat{f}(\xi) e^{-i\xi x} d\xi$$

where $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(y) e^{-iy\xi} dy$.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{|x| \to \infty} f(x) = 0$. Show that

$$\lim_{n \to \infty} T_n f(x) = f(x)$$

and that the convergence is uniform on $\mathbb{R}$.

(iii) Does the conclusion in (ii) hold if $f$ is merely in $L^\infty(\mathbb{R})$?

8. Let $\{f_n\}_{n=-\infty}^\infty$ be a sequence of continuous functions defined on the interval $[0, 1]$. Assume that for any $g \in L^\infty([0, 1])$:

$$\int_0^1 f_n(x) g(x) dx \to \int_0^1 g(x) dx,$$

as $n$ tends to $+\infty$.

(i) Show that there exists $C > 0$ such that: For every $n \in \mathbb{N}$, $\int_0^1 |f_n(x)| dx \leq C$. Show that necessarily $C \geq 1$.

(ii) Does $\int_0^1 |f_n(x)| dx$ tend to 1, as $n$ tends to $\infty$?

(iii) Does there exist a constant $A$ such that $\int_0^1 |f_n(x)|^2 dx \leq A$ for every $n \in \mathbb{N}$?

9. Let $f \in L^2([0, 1])$, $K \in L^2([0, 1] \times [0, 1])$ and define

$$T f(x) = \int_0^1 K(x, y) f(y) dy.$$ 

(**) Show that (**) defines $T f(x)$ for almost every $x \in [0, 1]$.

(ii) Show that $T$ is bounded on $L^2([0, 1])$.

(iii) Show that $T : L^2([0, 1]) \to L^2([0, 1])$ is a compact operator.
Problem I Define an infinite sequence of real numbers \( \{a_1, a_2, \ldots, a_n, \ldots\} \) by setting \( a_1 = 1 \), \( a_2 = 2 \), and \( a_{n+1} = 2a_n + 3a_{n-1} \) for \( n \geq 2 \).

(a) Let \( b_n = \frac{a_{n+1}}{a_n} \) for \( n \geq 1 \). Prove that \( \lim_{n \to \infty} b_n \) exists and evaluate the limit.

(b) What is the radius of convergence \( \rho \) of the infinite series \( \sum_{n=1}^{\infty} a_n x^n \)?

(c) For \( |x| < \rho \), evaluate \( \sum_{n=1}^{\infty} a_n x^n \). Does this infinite series converge when \( x = \rho \), the radius of convergence found in part (b)?

Problem II In this problem suppose that \( A \) and \( B \) are strictly positive real numbers.

(a) Show that there is a constant \( K > 0 \) so that for all \( A \) as above

\[
\int_0^{+\infty} \frac{dx}{A^3 + x^3} = K A^{-2}.
\]

Without computing \( K \) explicitly, show that \( K < 3/2 \).

(b) Show that there is a universal constant \( K \) so that for all \( A \) and \( B \) as above,

\[
\int_0^{+\infty} \frac{dx}{(A^3 + x^3)(B^3 + x^3)} \leq K (A + B)^{-3} \left[ \min\{A, B\} \right]^{-2}.
\]

(c) Find an estimate for

\[
\left| \int_0^{+\infty} \frac{\sin(x)}{(A + x)^3} \, dx \right|
\]

which is better as \( A \to 0 \) than can be obtained by observing that

\[
\left| \int_0^{+\infty} \frac{\sin(x)}{(A + x)^3} \, dx \right| \leq \int_0^{+\infty} \frac{|\sin(x)|}{(A + x)^3} \, dx \leq \int_0^{+\infty} \frac{dx}{A^3 + x^3} = K A^{-2}.
\]

Problem III

(a) Let \( \Omega \) be a convex set in \( \mathbb{R}^2 \) with smooth boundary. Using only the Fundamental Theorem of Calculus for functions of one variable, prove that if \( f \) and \( g \) are continuously differentiable functions in a neighborhood of the closure of \( \Omega \), then

\[
\oint_C f(x, y) \, dx + g(x, y) \, dy = \iint_{\Omega} \left[ \frac{\partial g}{\partial x}(x, y) - \frac{\partial f}{\partial y}(x, y) \right] \, dx \, dy
\]

where \( C \) is the simple closed curve bounding \( \Omega \) taken in the counter-clockwise direction.

(b) Evaluate

\[
\oint_C \frac{x y^2 \, dx - x^2 y \, dy}{(x^2 + y^2)^2}
\]

where \( C \) is the ellipse \( 25(x - 1)^2 + 16(y - 2)^2 = 400 \), taken in the counter-clockwise sense.
Problem IV  
(a) Give an example of a sequence of functions $f_n \in L^1([0,1])$, $n = 1, 2, \ldots$ and a function $g \in L^1([0,1])$ with the following properties:

1. $f_n(x) \to g(x)$ for almost all $x \in [0,1]$;  
2. $\int_0^1 |f_n(x)| \, dx = 2$ for every $n = 1, 2, \ldots$;  
3. $\int_0^1 |g(x)| \, dx = 1$.

(b) Show that for any sequence $\{f_n\}$ and function $g$ as in part (a) it follows that 
\[
\lim_{n \to \infty} \int_0^1 |f_n(x) - g(x)| \, dx = 1.
\]

Problem V  
Let $f \in L^4([0,1])$.

(a) Show that $f \in L^4([0,1])$ and that $\|f\|_2 \leq \|f\|_4$.

(b) Does there exist a constant $C$ so that for all $f \in L^4([0,1])$, $\|f\|_4 \leq C \|f\|_2$?

(c) For a given function $f_0 \in L^4([0,1])$, let $C$ be a constant such that 
\[
\int_0^1 |f_0(x)|^4 \, dx \leq C \left( \int_0^1 |f_0(x)|^2 \, dx \right)^2.
\]
Find a constant $A$ depending only on $C$ so that 
\[
\left( \int_0^1 |f_0(x)|^2 \, dx \right)^2 \leq A \int_0^1 |f_0(x)| \, dx.
\]
(HINT: Estimate $\int_0^1 |f_0(x)|^2 \, dx = \int_0^1 |f_0(x)|^\alpha |f_0(x)|^{2-\alpha} \, dx$ by using Hölder's inequality with appropriate exponents and an appropriate $\alpha < 1$.)

Problem VI  
Define a function $\phi$ on $\mathbb{R}$ by setting 
\[
\phi(x) = \begin{cases} 
1 - |x| & \text{if } |x| < 1, \\
0 & \text{if } |x| \geq 1.
\end{cases}
\]

(a) Show that if $g$ is a continuously differentiable function on $\mathbb{R}$, then the convolution of $g$ with $\phi$ is also continuously differentiable.

(b) Find a function $\chi \in L^\infty(\mathbb{R})$ so that if $g$ is continuously differentiable on $\mathbb{R}$, then 
\[
\frac{d(g \ast \phi)}{dx} (x) = g \ast \chi(x).
\]

(c) Show that if $f \in L^1(\mathbb{R})$, then the convolution of $f$ with $\phi$ is continuously differentiable.
(HINT: Approximate $f$ by a sequence of continuously differentiable functions $g_n$, and then use (a) and (b) and results about limits of continuously differentiable functions.)
Problem VII. Let $0 < \alpha < 1$. Evaluate the improper integral
\[ \int_0^{+\infty} \frac{dx}{x^\alpha (1 + x)}. \]

(HINT: Consider the complex plane slit along the positive real axis, and consider a closed contour in this slit plane consisting of a part of a large circle of radius $R$ taken in the counterclockwise sense, a part of a small circle of radius $\epsilon$ taken in the clockwise sense, and two lines parallel to the positive axis joining these circles, one above the positive $x$ axis and one below.)

Problem VIII. Let $f$ be a holomorphic function defined in the unit disc $D$. Show that the following two assertions are equivalent:

1. There exists a constant $C > 0$ and a positive integer $n$ such that
\[ |f(z)| \leq \frac{C}{(1 - |z|^2)^n}. \]

2. There exists a positive constant $A$ and a positive integer $k$ such that the coefficients \( \{a_n\} \) in the power series expansion \( f(z) = \sum_{m=0}^{\infty} a_m z^m \) satisfy the inequality
\[ |a_m| \leq A m^k. \]

Try to give sharp results relating the constants $C$ and $n$ to the constants $A$ and $k$.

Problem IX

(a) Let $f$ be a holomorphic function defined in the unit disc $D$ and suppose that $f(0) = 1$ and $Re[f(z)] > 0$ for all $z \in D$. Show that for $-1 < x < +1$
\[ |f(x)| \leq \frac{1 + |x|}{1 - |x|}. \]

What can you say if there is equality at some point $x \neq 0$?

(b) Let $V = \{z = re^{i\theta} \in \mathbb{C} \mid r > 0 \text{ and } |\theta| < \frac{\pi}{4}\}$. Let $D$ denote the unit disc, and suppose $f : D \to V$ is holomorphic with $f(0) = 1$. Prove that if $-1 < x < +1$ then
\[ |f(x)| \leq \left(\frac{1 + |x|}{1 - |x|}\right)^{\frac{1}{2}}. \]
**Problem I**  Let \( \{a_n\} \) be a sequence of positive real numbers. Assume that for every \( n \geq 1 \)
\[
\frac{a_{n+1}}{a_n} \leq 1 + \frac{1}{n^2}.
\]
a) If \( a_{n+1}/a_n \) does not tend to 0 as \( n \to \infty \), prove that
\[
\lim_{n \to \infty} a_n = 0.
\]
b) Discuss what can happen if \( \lim_{n \to \infty} a_{n+1}/a_n = 1 \).

**Problem II**
a) Show that
\[
\inf \left\{ \int_0^1 |u'(t)|^2 \, dt \right\} = 1
\]
where the infimum is taken over all real valued functions \( u \in C^1([0, 1]) \) with \( u(0) = 0 \) and \( u(1) = 1 \).
b) Compute
\[
\inf \left\{ \int_0^1 |u'(t)|^2 \, t \, dt \right\}
\]
where the infimum is taken over the same class of functions as in part a). (Hint: Although \( u(t) = t^\alpha \) is not in \( C^1([0, 1]) \) for \( 0 < \alpha < 1 \), first try \( u(t) = t^\alpha \).)
c) Compute
\[
\inf \left\{ \iint_B |\nabla u(x, y)|^2 \, dx \, dy \right\}
\]
where the infimum is taken over all functions continuously differentiable on \( B \), the closed unit disk, which satisfy \( u(0, 0) = 0 \) and \( u|_{\partial B} = 1 \).

**Problem III** Let \( \{a_n\} \) be a sequence of complex numbers such that \( \sum_{n=-\infty}^{+\infty} |a_n| < +\infty \). Define a function \( f \) by the infinite series
\[
f(x) = \sum_{n=-\infty}^{+\infty} a_n e^{i n x}.
\]
a) Prove that \( f \) is a continuous function for \( x \in \mathbb{R} \), that \( f(x + 2\pi) = f(x) \) for all \( x \), and that and that
\[
\frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-i n t} \, dt = a_n.
\]
b) Prove that if \( f \in C^2(\mathbb{R}) \), there is a constant \( C \) so that
\[
|a_n| \leq \frac{C}{n^2}.
\]
c) Let \( I(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \, dt \) and for \( N \) a positive integer, let \( I_N(f) \) be the numerical approximation to \( I(f) \) given by
\[
I_N(f) = \frac{1}{N} \sum_{k=0}^{N-1} f\left( \frac{2k\pi}{N} \right).
\]
If \( f \in C^2(\mathbb{R}) \), estimate \( |I(f) - I_N(f)| \) as \( N \to \infty \).
Problem IV Let \(\phi\) be a continuously differentiable function, with compact support, defined on \(\mathbb{R}\).

a) Is the integral
\[
\iiint_{\mathbb{R}^+ \times \mathbb{R}^+} \phi(xt) \sin x \, dx \, dt
\]
convergent?

b) Give a meaning to each side of the equality below, and (possibly at the same time) justify that equality holds:
\[
\int_0^{+\infty} \left( \int_0^{+\infty} \phi(xt) \sin x \, dx \right) \, dt = \int_0^{+\infty} \left( \int_0^{+\infty} \phi(xt) \sin x \, dx \right) \, dt.
\]
(Hint: In order to estimate \(\int_0^{+\infty} \left( \int_0^{+\infty} \phi(xt) \sin x \, dx \right) \, dt\), first show that
\[
\left| \int_0^{+\infty} \phi(xt) \sin x \, dx \right| \leq \sup \{ |\phi| \} + \int |\phi'|.
\]
Then notice this integration in \(t\) needs to be done only on the interval \(0 \leq t \leq A/R\) if \(\phi \equiv 0\) off the interval \([-A,+A]\).)

Problem V Let \(\mathcal{C}\) denote the space of continuous functions on \(\mathbb{R}\) which are \(2\pi\) periodic. Let \(\mathcal{C}^\infty\) denote the subspace of functions which are infinitely differentiable. For \(f\) and \(g\) in \(\mathcal{C}\), their convolution is defined by
\[
f \ast g(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y) g(x - y) \, dy.
\]
Prove that the following two conditions are equivalent:

\(C_1\): The function \(f \in \mathcal{C}^\infty\).

\(\bar{C}_1\): For every \(g \in \mathcal{C}\), the convolution \(f \ast g \in \mathcal{C}^\infty\).

Problem VI Let \(f\) be a function defined and bounded on the set \(Q = \{(x,y) \in \mathbb{R}^2 \mid |x| \leq 1, |y| \leq 1\}\). Suppose
1. for each fixed \(y \in [-1,1]\) the function \(x \rightarrow f(x,y)\) is measurable;
2. for all \((x,y) \in Q\), \(\frac{\partial f}{\partial y}(x,y)\) exists;
3. the function \(\frac{\partial f}{\partial y}(x,y)\) is bounded on \(Q\).

Let \(F(y) = \int_{-1}^{+1} f(x,y) \, dx\). Prove that \(F\) is differentiable, and that
\[
F'(y) = \int_{-1}^{+1} \frac{\partial f}{\partial y}(x,y) \, dx.
\]
Problem VII (Math 722) Let $f$ be holomorphic on the unit disk $D$, and suppose $f(0) = 0$ and $f'(0) = 1$. Assuming that the function $f : D \to \mathbb{C}$ is one to one, prove that the following four conditions are equivalent:
(i) $f(D) = D$.
(ii) $f(D) \subseteq D$.
(iii) $f(D) \supset D$.
(iv) The area of $f(D)$ equals $\pi$.

Problem VIII (Math 722)

a) Show that there is a unique holomorphic function $f$ defined on the set $\{z \in \mathbb{C} | |z| > 1 \}$ such that for $|z| > 1$

$$
(f(z))^2 = z^{10} + 1, \quad \text{and} \quad f(2) > 0.
$$

b) If the function in part a) is denoted by $\sqrt{z^{10} + 1}$, evaluate

$$
\int_{|z|=2} \frac{z^4}{\sqrt{z^{10} + 1}} dz.
$$

Problem IX (Math 722) Let $f$ be a holomorphic function on the unit disk $D$, and assume that there is a constant $M$ so that for $z \in D$,

$$
|f(z)| \leq \frac{M}{(1 - |z|)^2}.
$$

Find an integer $k$ so that if $\varphi$ is any $k$-times continuously differentiable complex valued function on $\mathbb{R}$, then

$$
\lim_{r \to 1^-} \left( \int_0^{2\pi} f(r e^{i\theta}) \varphi(e^{i\theta}) d\theta \right)
$$

exists.
Problem VII (Math 725) Let $\delta_0$ denote the Dirac mass at the origin $0 \in \mathbb{R}$.
a) Define the derivates $\delta_0^{(k)}$ in the sense of distributions.
b) Find bounded functions $f$ and $g$ on $\mathbb{R}$ so that (in the sense of distributions)

\[ f'' - f = \delta_0; \]
\[ g'' - g = \delta'_0. \]

c) Is there a bounded function $h$ on $\mathbb{R}$ so that (in the sense of distributions)

\[ h'' - h = \delta''_0? \]

Problem VIII (Math 725) Let $S$ be a subset of $\mathbb{R}^n$. Let

\[ \tilde{S} = \{ x \in \mathbb{R}^n \mid \forall y \in S, x \cdot y > 0 \}. \]

a) Prove that $\tilde{S}$ is a convex cone in $\mathbb{R}^n$.
b) Let $\tilde{v}_1, \ldots, \tilde{v}_{n+1}$ be non-zero vectors in $\mathbb{R}^n$. Assume that no collection of $n$ of these vectors is linearly dependent. Let $S = \{ \tilde{v}_1, \ldots, \tilde{v}_{n+1} \}$. Show that one can write

\[ \bar{v} = \alpha_1 \tilde{v}_1 + \cdots + \alpha_{n+1} \tilde{v}_{n+1} \]

with all $\alpha_j > 0$ if and only if $\tilde{S} = \emptyset$.

Problem IX (Math 725) Let $E$ be the space of functions $f \in L^2(\mathbb{R}^2)$ whose partial derivative \[ \frac{\partial f}{\partial y} \] (in the sense of distributions) also belongs to $L^2(\mathbb{R}^2)$.
a) Define a "natural" scalar product on $E$ such that with this product, $E$ is a Hilbert space. In particular, show that $E$ is complete. Show that $C_0^\infty(\mathbb{R}^2)$ is dense in $E$.
b) Show be an example that $E \not\subseteq L^2_{loc}(\mathbb{R}^2)$. But also show that for any $f \in E$, it makes sense to speak about the restriction of $f$ to the line $\mathbb{R} \times \{0\}$.
Problem I. For \( n \geq 0 \) let \( a_n = [\log(2 + n)]^{-1} \).
1. For which complex numbers \( z \) does the series \( \sum_{n=0}^{\infty} a_n z^n \) converge?
2. For which complex numbers \( z \) does the series \( \sum_{n=0}^{\infty} a_n z^n \) converge absolutely?
3. On which compact sets of the complex plane does the series \( \sum_{n=0}^{\infty} a_n z^n \) converge uniformly?

Problem II.
1. What is the volume of the region \( \Omega \) in \( \mathbb{R}^n \), defined by
   \( \Omega = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j > 0, \quad 0 < x_1 + x_2 + \cdots + x_n < 1\} \)?
2. What is the area of the parallelogram spanned by the vectors \((1, 1, -1, 1)\) and \((2, 1, 2, 1)\) in \( \mathbb{R}^4 \)?
3. What is the (3 dimensional) volume of the box (parallelepiped) spanned by the vectors \((1, 1, 0, 0, 0), (0, 1, 1, 0), \) and \((0, 0, 1, -1, 1)\) in \( \mathbb{R}^5 \)?

Problem III.
1. Let \( \psi \in C_0(\mathbb{R}) \) be a compactly supported continuous function. Show that
   \[ \lim_{N \to \infty} \frac{1}{N} \int_0^{\infty} \frac{\psi(x/N)}{\sqrt{1 + x}} \, dx = 0. \]
2. Let
   \[ J_N = \int_0^N e^{ix} \sqrt{1 + x} \, dx. \]
   Does \( \lim_{N \to -\infty} J_N \) exist?
3. Let \( \chi \in C_0^2(\mathbb{R}) \). Prove that
   \[ \lim_{N \to -\infty} \int_0^{\infty} \chi(\frac{x}{N}) e^{ix} \sqrt{1 + x} \, dx \]
   exists.
4. To what extent does the limit in part (3) depend on the choice of the function \( \chi \)?

Problem IV. Let \( \{r_k \mid k = 1, 2, \ldots\} \), be an enumeration of the set of rational numbers in the interval \([0, 1]\). For \( x \in [0, 1] \) set
   \[ F(x) = \sum_{j=1}^{\infty} \frac{2^{-k}}{\sqrt{|e^{2\pi i x} - e^{2\pi ir_j}|}}. \]
1. Show that \( F \) is unbounded in a neighborhood of every point of \([0, 1]\).
2. Show that \( F(x) \) is finite for almost all \( x \in [0, 1] \).
3. For which \( p \) is it true that \( F \in L^p([0, 1]) \)?
Problem V. For $f \in L^2(\mathbb{R}_+)$ define

$$Tf(x) = \int_0^\infty f(y) \frac{dy}{x + 2y}.$$ 

Show that $T$ is a bounded operator on $L^2(\mathbb{R}_+)$.

Problem VI. Let $0 \leq f \in L^1(\mathbb{R}_+)$, and define

$$\omega_f(y) = \left| \{ x \in (0, +\infty) \mid f(x) > y \} \right|.$$ 

(Here, for any set $E \subset (0, +\infty)$, $|E|$ denotes the Lebesgue measure of $E$.)

1. Prove that for any $f \in L^1(\mathbb{R}_+)$, $\omega_f(y)$ is a non-increasing function of $y$, and that

$$\lim_{y \rightarrow +\infty} \omega_f(y) = 0.$$ 

If $f > 0$ almost everywhere, show that

$$\lim_{y \rightarrow 0^+} \omega_f(y) = +\infty$$

2. Prove that for any $p > 0$

$$\int_0^{+\infty} f(x)^p \, dx = p \int_0^{+\infty} y^{p-1} \omega_f(y) \, dy.$$ 

3. If $f$ is the function defined by $f(x) = x(1-x)$ if $0 \leq x \leq 1$, and $f(x) = 0$ if $x > 1$, determine the non-increasing function $f^*$, defined on $\mathbb{R}_+$ with the property

$$\omega_f(y) = \omega_{f^*}(y), \quad \forall y > 0$$

($f^*$ is called the non-increasing rearrangement of $f$).

Problem VII. Let $f(x) = e^{2x} \cos e^x$. Then $f$ defines a distribution in $\mathcal{D}'(\mathbb{R})$. Prove or disprove that this distribution is tempered.
Problem VIII. In what follows you are given examples of subsets $Y$ of Hilbert spaces $H$. Prove or disprove in each instance that $Y$ is compact in $H$.

1. $H = L^2([−π, π])$, $Y$ is the set of all $f \in H$ whose Fourier coefficients

$$\hat{f}_k \overset{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt$$

satisfy

$$\sum_{k=-\infty}^{\infty} k^2 |\hat{f}_k|^2 \leq 1.$$

2. $H = L^2(\mathbb{R})$, $Y$ is the set of all $f \in H$ whose Fourier transforms satisfy

$$\int (1 + |\xi|)^{\frac{1}{2}} |\hat{f}(\xi)|^2 \, d\xi \leq 1.$$

3. $H = L^2([0, 1])$, and $Y$ is the set of all continuous functions satisfying $f(1) = 1$ and $|f(x) − f(x')| \leq |x − x'|^{\frac{1}{5}}$ for all $x, x' \in [0, 1]$.

Problem IX.

1. Show that for all $k = 1, 2, \ldots, n$

$$\langle T_k, \varphi \rangle \overset{\text{def}}{=} \lim_{\epsilon \to 0} \int \varphi(x) \frac{x_k}{|x|^{n+1}} \, dx_1 \ldots dx_n$$

defines a tempered distribution $T_k \in \mathcal{S}'(\mathbb{R})$.

2. Prove that, in the sense of distributions, one has

$$\frac{\partial}{\partial x_k} \left( \frac{1}{|x|^{n-1}} \right) = -(n - 1) T_k.$$

3. Given that the Fourier transform of $\frac{1}{|x|^{n-1}}$ is $C_n |\xi|^{\frac{n}{2}}$, find the Fourier transform of $T_k$. 

Problem V. For $f \in L^2(\mathbb{R}_+)$ define

$$Tf(x) = \int_0^\infty f(y) \frac{dy}{x+2y}.$$  

Show that $T$ is a bounded operator on $L^2(\mathbb{R}_+)$. 

Problem VI. Let $0 \leq f \in L^1(\mathbb{R}_+)$, and define

$$\omega_f(y) = \left| \{ x \in (0, +\infty) \mid f(x) > y \} \right|.$$  

(Here, for any set $E \subseteq (0, +\infty)$, $|E|$ denotes the Lebesgue measure of $E$.)

1. Prove that for any $f \in L^1(\mathbb{R}_+)$, $\omega_f(y)$ is a non-increasing function of $y$, and that

$$\lim_{y \to +\infty} \omega_f(y) = 0.$$  

If $f > 0$ almost everywhere, show that

$$\lim_{y \to 0} \omega_f(y) = +\infty$$

2. Prove that for any $p > 0$

$$\int_0^{+\infty} f(x)^p \, dx = p \int_0^{+\infty} y^{p-1} \omega_f(y) \, dy.$$  

3. If $f$ is the function defined by $f(x) = x(1-x)$ if $0 \leq x \leq 1$, and $f(x) = 0$ if $x > 1$, determine the non-increasing function $f^*$, defined on $\mathbb{R}_+$ with the property

$$\omega_f(y) = \omega_f^*(y), \quad \forall y > 0$$

($f^*$ is called the non-increasing rearrangement of $f$).

Problem VII. Suppose that $f(z) = \sum_0^\infty a_n z^n$ is holomorphic in $\mathbb{D}$ and that there is a constant $A$ so that $|f'(z)| \leq \frac{A}{1-|z|^2}$ for all $z$ in $\mathbb{D}$.

1. Show that there is a constant $B$ such that $|a_n| \leq B$ for all $n$.

2. Is the converse true? That is, if $|a_n| \leq B$ for all $n$ and $f(z) = \sum_0^\infty a_n z^n$, does it necessarily follow that $|f'(z)| \leq \frac{A}{1-|z|^2}$ for some constant $A$?
Problem VIII. Let \( \mathcal{B} \) denote the set of holomorphic functions on \( \mathbb{D} \) such that \( |f(z)| \leq 1 \) for all \( |z| < 1 \).

1. Find
\[
\sup \left\{ |f(0)| : f \in \mathcal{B} \text{ and } f\left(\frac{i}{2}\right) = f\left(\frac{i}{2}\right) = f\left(\frac{-1}{2}\right) = f\left(\frac{-i}{2}\right) = 0 \right\}.
\]

2. Find
\[
\sup \left\{ |f\left(\frac{1}{2}\right) + f\left(\frac{-1}{2}\right)| : f \in \mathcal{B} \text{ and } f(0) = 0 \right\}.
\]

Problem IX.

1. Show that if \( f \) is holomorphic on \( \mathbb{C} \) and \( f(z + m + ni) \equiv f(z) \) for all integers \( m \) and \( n \) then \( f \) is constant.

2. Show that if \( f \) is meromorphic and non-constant on \( \mathbb{C} \) and \( f(z + m + ni) \equiv f(z) \) for all integers \( m \) and \( n \) then it cannot be true that on the set \( Q = \{ x + iy : 0 \leq x < 1, \ 0 \leq y < 1 \} \), \( f \) has a pole of order 1 at some point and is otherwise holomorphic on a neighborhood of every other point of \( Q \). Hint: if \( f \) has no pole on \( \partial Q \), evaluate \( \int_{\partial Q} f(z)dz \).
Problem I  Let \( \{a_n\} \), \( n = 1, 2, \ldots \), be a sequence of strictly positive real numbers.

a) Prove that
\[
\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \to \infty} \sqrt[n]{a_n} \leq \limsup_{n \to \infty} \sqrt[n]{a_n} \leq \limsup_{n \to \infty} \frac{a_{n+1}}{a_n}.
\]

b) Give an example of a sequence \( \{a_n\} \) of strictly positive real numbers such that all of the above inequalities are strict inequalities.

Problem II  Let
\[ Q = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < x < 1, \ 0 < y < 1 \right\}. \]

For positive real numbers \( a \) and \( b \) define a function \( F_{a,b} : Q \to \mathbb{R} \) by the equation
\[
F_{a,b}(x, y) = x^a y^b \int_0^\infty \frac{ds}{(x + s)(y^3 + s^3)}.
\]

For which \( a \) and \( b \) is it true that \( F_{a,b} \) is a bounded function on \( Q \)?

Problem III  Let \( U = \{ x \in \mathbb{R}^n \mid |x| < 1 \} \) be the open unit ball in \( \mathbb{R}^n \) and let \( \rho : U \to \mathbb{R} \) be a \( C^\infty \) function such that \( \rho(0) = 0 \) and \( \nabla \rho(0) \neq 0 \). Let \( \Sigma = \{ x \in U \mid \rho(x) = 0 \} \). For \( x \in U \) let
\[ d(x) = \inf_{y \in \Sigma} |x - y|. \]

a) For \( x \in V = \{ x \in \mathbb{R}^n \mid |x| < \frac{1}{2} \} \), prove that there is a point \( y \in \Sigma \) such that \( d(x) = |x - y| \).

b) For \( x \in V - \Sigma \) and for any \( y \in \Sigma \) such that \( d(x) = |x - y| \), prove that vector \( \nabla \rho(y) \) is a scalar multiple of the vector \( x - y \).

c) Prove that there is an open set \( W \) with \( 0 \in W \subseteq V \) and a \( C^\infty \) function \( \varphi : W \to \mathbb{R} \) so that for \( x \in W \), \( |\varphi(x)| = d(x) \).

Problem IV  Let \( \mathcal{B} \) denote the set of all Borel subsets of \( \mathbb{R} \). Let \( \mu : \mathcal{B} \to [0, \infty) \) (the finite, non-negative real numbers) be a set function with the property that if \( E_j \in \mathcal{B} \) are mutually disjoint, then
\[
\mu \left( \bigcup_{j=1}^\infty E_j \right) = \sum_{j=1}^\infty \mu(E_j).
\]

a) Prove that if \( \{ E_j \} \) is a sequence of Borel sets with \( E_j \subseteq E_{j+1} \), then
\[
\mu \left( \bigcup_{j=1}^\infty E_j \right) = \lim_{j \to \infty} \mu(E_j).
\]

b) Prove that if \( \{ E_j \} \) is a sequence of Borel sets with \( E_j \supset E_{j+1} \), then
\[
\mu \left( \bigcap_{j=1}^\infty E_j \right) = \lim_{j \to \infty} \mu(E_j).
\]

\c) Suppose that for every Borel set \( E \) with Lebesgue measure \( |E| = 0 \), it follows that \( \mu(E) = 0 \). Prove that for every \( \epsilon > 0 \) there exists \( \delta > 0 \) so that if \( E \in \mathcal{B} \) and \( |E| < \delta \), then \( \mu(E) < \epsilon \).
Problem V  Let $f \in L^1(\mathbb{R})$. For $x \in \mathbb{R}$ and $y > 0$ set

$$F(x, y) = \frac{1}{\sqrt{y}} \int_{\mathbb{R}} f(t) e^{-\pi \frac{(x-t)^2}{y}} \, dt.$$ 

a) Show that $\lim_{y \to \infty} F(x, y)$ exists for all $x \in \mathbb{R}$. What is the limit?
b) Show that there is a constant $C$ independent of $f$ so that for all $y > 0$

$$|F(x, y)| \leq C \sup_{r > 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(t)| \, dt.$$ 
c) Show that $\lim_{y \to 0} F(x, y)$ exists for almost all $x \in \mathbb{R}$. What is the limit?

Problem VI
a) Prove Hölder’s inequality for measurable functions $f$ and $g$ defined on $\mathbb{R}$: if $1 < p, q < +\infty$ and if $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\int_{\mathbb{R}} |f(x)g(x)| \, dx \leq \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}} |g(x)|^q \, dx \right)^{\frac{1}{q}}.$$ 
b) Suppose that $f$ is a measurable function on $\mathbb{R}$ and that $|f| \neq 0$ on a set of positive measure. For $0 < p < +\infty$, let

$$A_f(p) = \log \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right),$$

and let

$$E_f = \left\{ p \in (0, \infty) \, | \, A_f(p) < +\infty \right\}.$$ 

Using part a), prove that $E_f$ is connected and prove that if $E_f$ has non-empty interior, then $A_f$ is a convex function on the interior of $E_f$.
c) Give an example of a function $f$ for which $E_f$ in part b) consists of exactly one point.

Problem VII  State carefully and give a complete proof of the Baire Category Theorem. Then describe one significant application of this result.

Problem VIII
a) If $T$ is a distribution on $\mathbb{R}^2$, give a precise definition of the distribution $\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$.
b) For $(x, y) \in \mathbb{R}^2$, let $F(x, y) = \log(x^2 + y^2)$. Derive an explicit formula for $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$ in the sense of distributions.

Problem IX  For $f \in L^1(\mathbb{R}^n)$, the Fourier transform of $f$ is the function

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) \, dx.$$ 
a) Prove that if $f \in L^1(\mathbb{R}^n)$ then $\hat{f}$ is a continuous function on $\mathbb{R}^n$, and that $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$.
b) Suppose that $f$ and $g$ belong to $L^1(\mathbb{R}^n)$ and that there is a real number $\alpha > \frac{n}{2}$ such that

$$\hat{f}(\xi) = (1 + |\xi|^2)^{\alpha} \hat{g}(\xi).$$

Prove that, after redefining $g$ on a set of measure zero if necessary, $g$ is continuous on $\mathbb{R}^n$. 
Problem V  Let $f \in L^1(\mathbb{R})$. For $x \in \mathbb{R}$ and $y > 0$ set
\[ F(x, y) = \frac{1}{\sqrt{y}} \int_{\mathbb{R}} f(t) e^{-\pi \frac{(x-t)^2}{y}} \, dt. \]

a) Show that $\lim_{y \to \infty} F(x, y)$ exists for all $x \in \mathbb{R}$. What is the limit?

b) Show that there is a constant $C$ independent of $f$ so that for all $y > 0$
\[ |F(x, y)| \leq C \sup_{r > 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(t)| \, dt. \]

c) Show that $\lim_{y \to 0} F(x, y)$ exists for almost all $x \in \mathbb{R}$. What is the limit?

Problem VI

a) Prove Hölder’s inequality for measurable functions $f$ and $g$ defined on $\mathbb{R}$ if $1 < p, q < +\infty$ and if $\frac{1}{p} + \frac{1}{q} = 1$ then
\[ \int_{\mathbb{R}} |f(x)g(x)| \, dx \leq \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}} |g(x)|^q \, dx \right)^{\frac{1}{q}}. \]

b) Suppose that $f$ is a measurable function on $\mathbb{R}$ and that $|f| \neq 0$ on a set of positive measure. For $0 < p < +\infty$, let
\[ A_f(p) = \log \left[ \int_{\mathbb{R}} |f(x)|^p \, dx \right], \]
and let
\[ E_f = \left\{ p \in (0, \infty) \mid A_f(p) < +\infty \right\}. \]
Using part a), prove that $E_f$ is connected and prove that if $E_f$ has non-empty interior, then $A_f$ is a convex function on the interior of $E_f$.

c) Give an example of a function $f$ for which $E_f$ in part b) consists of exactly one point.

Problem VII  Let $f$ be a holomorphic function defined in an open set $\Omega$ containing the origin, and suppose there is an integer $m \geq 1$ so that $f^{(j)}(0) = 0$ for $0 \leq j \leq m - 1$ and $f^{(m)}(0) \neq 0$. Prove that there are constants $A$ and $B$ so that for $0 < |w| < A$, the equation $f(z) = w$ has exactly $m$ distinct roots in the set $\{ z \in \mathbb{C} \mid |z| < B \}$.

Problem VIII  For each of the following, prove that there is no holomorphic function $f$ defined on the unit disk $\mathbb{D}$ with the stated properties:

a) \[ \lim_{|z| \to 1^{-}} |f(z)| = +\infty. \]

b) For all $z \in \mathbb{D}$, $|f(z)| < 1$ and
\[ f \left( \frac{1}{2} \right) = 0, \quad f \left( \frac{1}{3} \right) = 0, \quad f(0) = \frac{1}{5}. \]

Problem IX  Let $\alpha$ be a real number. Evaluate $\int_{-\infty}^{+\infty} \frac{\cos(\alpha x)}{e^x + e^{-x}} \, dx$. 

Problem I  Let \( \{a_n\} \), \( n \geq 0 \), be an infinite sequence of complex numbers. Recall that the infinite series
\[
\sum_{n=0}^{\infty} a_n
\]
converges to a complex number \( S \) if and only if the sequence of partial sums \( \{S_N = \sum_{n=0}^{N} a_n\} \)
converges to \( S \). The infinite series \( \sum_{n=0}^{\infty} a_n \) is said to be Abel summable to a complex number \( A \) if for all real numbers \( x \) with \( 0 \leq x < 1 \), the infinite series \( \sum_{n=0}^{\infty} a_n x^n \) converges, and \( \lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = A \).

(a) Suppose that \( \sum_{n=0}^{\infty} a_n \) converges to \( S \). Prove that \( \sum_{n=0}^{\infty} a_n \) is Abel summable to \( S \).

(b) Give an example that shows that a series \( \sum_{n=0}^{\infty} a_n \) can be Abel summable, but can fail to converge.

(c) Prove that if \( a_n \geq 0 \) and if \( \sum_{n=0}^{\infty} a_n \) is Abel summable to a complex number \( A \), then the infinite series \( \sum_{n=0}^{\infty} a_n \) converges to \( A \).

Problem II  Let \( \{f_n\} \), \( n \geq 1 \), be an infinite sequence of real valued continuous functions defined for all real numbers \( x \in \mathbb{R} \). Suppose there is a function \( g \) so that \( f_n \) converges uniformly to \( g \) on \( \mathbb{R} \).

(a) Either prove that \( f_n^2 \) converges uniformly to \( g^2 \) on \( \mathbb{R} \), or give a counter-example to this assertion.

(b) Suppose that \( \varphi \) is a uniformly continuous function defined on \( \mathbb{R} \). Prove that \( \varphi(f_n) \) converges uniformly to \( \varphi(g) \) on \( \mathbb{R} \).

(c) Suppose that \( \varphi \) is a bounded continuous function defined on \( \mathbb{R} \). Either prove that \( \varphi(f_n) \) converges uniformly to \( \varphi(g) \), or give a counter-example to this assertion.

(d) Suppose that each \( f_n \) is continuously differentiable, and that \( f'_n \) converges uniformly to a function \( h \). Prove directly from the definitions that \( g \) is differentiable, and that \( g'(x) = h(x) \) for all \( x \in \mathbb{R} \).

Problem III  Let \( \varphi \) be an infinitely differentiable function on \( \mathbb{R}^3 \) with compact support. Recall that the Laplacian is the second order differential operator \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \).

(a) For which \( p > 0 \) does \( \lim_{\epsilon \to 0} \iiint_{|x^2+y^2+z^2|>\epsilon^2} \frac{\varphi(x, y, z)}{|x^2+y^2+z^2|^{p/2}} \, dx \, dy \, dz \) exist for all such functions \( \varphi \)?

(b) For which \( p > 0 \) is it true that \( \Delta \left[ \left( x^2 + y^2 + z^2 \right)^{-p/2} \right] = 0 \) for \( (x, y, z) \neq (0, 0, 0) \)?

(c) Prove that \( \varphi(0) = \lim_{\epsilon \to 0} -\frac{1}{4\pi} \iiint_{|x^2+y^2+z^2|>\epsilon^2} \frac{\Delta \varphi(x, y, z)}{\sqrt{x^2+y^2+z^2}} \, dx \, dy \, dz \).
Problem IV Suppose \( \{f_n\} \) is a sequence of complex valued measurable functions on the interval \([0, 1]\) and suppose that \( \lim_{n \to \infty} f_n(x) = g(x) \) for almost every \( x \in [0, 1] \).

(a) Prove that \( g \) is a measurable function.

(b) Prove the following version of Egoroff's Theorem: Given any \( \epsilon > 0 \), there exists a measurable set \( E \subset [0, 1] \) with the Lebesgue measure \( |E| < \epsilon \) such that \( f_n \to g \) uniformly on \([0, 1]\)\(\setminus E\).

(You may use without proof the basic properties of Lebesgue measure, such as countable additivity. However, your proof should not depend on quoting results about convergence theorems for Lebesgue integrals.)

Problem V Let \( \{f_n\} \) be a sequence of functions belonging to \( L^1(\mathbb{R}) \), the set of integrable functions on the real line \( \mathbb{R} \). Suppose that there is a measurable function \( f \) so that \( \lim_{n \to \infty} f_n(x) = f(x) \) for almost every \( x \in \mathbb{R} \). Consider the following statements:

1. \( \lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)| \, dx = A \) exists.
2. \( \int_{\mathbb{R}} |f(x)| \, dx < +\infty \).
3. \( \lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)| \, dx = \int_{\mathbb{R}} |f(x)| \, dx < +\infty \).
4. \( \lim_{n \to \infty} \int_E |(f_n - f)(x)| \, dx = 0 \) for every measurable set \( E \subset \mathbb{R} \).

Discuss (with proofs or examples) which of these four statements do or do not imply other statements in the list.

Problem VI Let \( f \in L^1(\mathbb{R}) \). For \( y > 0 \), define \( f_y(x) = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{f(x - t) \, y^3}{(t^2 + y^2)^2} \, dt \).

(a) Prove that for each \( y > 0 \) the function \( f_y \in L^1(\mathbb{R}) \).

(b) Prove that \( \lim_{y \to 0} \|f - f_y\|_1 = 0 \). (You may use the fact that \( \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{(t^2 + 1)^2} = 1 \).)

(c) Prove that there is a sequence \( y_j \to 0 \) so that for almost all \( x \in \mathbb{R} \), \( \lim_{j \to \infty} f_{y_j}(x) = f(x) \).

(d) A stronger true statement than that in part (c) is that for almost all \( x \in \mathbb{R} \), \( \lim_{y \to 0} f_y(x) = f(x) \). Without giving proofs, discuss some of the steps that are involved in proving this result.
Problem VII Let $H$ be a real Hilbert space with inner product $(\cdot, \cdot)$. A sequence $\{x_n\}_{n \geq 1}$ converges weakly to $x_0$ if and only if $\lim_{n \to \infty} (x_n, y) = (x_0, y)$ for all $y \in H$. A sequence $\{x_n\}_{n \geq 1}$ converges strongly to $x_0$ if and only if $\lim_{n \to \infty} ||x_n - x_0|| = 0$. A bounded linear operator $T : H \to H$ is compact if for every bounded sequence $\{x_n\}$ of vectors in $H$, there is a subsequence of integers $n_j \to \infty$ so that $T(x_{n_j})$ converges strongly in $H$.

(a) Prove that if a sequence $\{x_n\}$ converges weakly to $x_0$ and if a sequence $\{y_n\}$ converges strongly to $y_0$, then $\lim_{n \to \infty} (x_n, y_n) = (x_0, y_0)$.

(b) Prove that if $T$ is a compact operator on the Hilbert space $H$, then there is a vector $x_0 \in H$ with $||x_0|| = 1$ and $\sup_{||z|| \leq 1} (T(z), z) = (T(x_0), x_0)$.

(c) Let $T$ be a compact operator on a Hilbert space $H$. Let $\lambda \in \mathbb{C}$ be a non-zero complex number, and let $E_\lambda = \{x \in H \mid (T(x), x) = \lambda \}$. Prove that $E_\lambda$ is a finite dimensional subspace of $H$.

Problem VIII A distribution $T$ on $\mathbb{R}^2$ is a continuous linear functional on the space $C^0_c(\mathbb{R}^2)$.

(a) Let $T$ be a distribution on $\mathbb{R}^2$. Give a precise definition of what it means that $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0$ on an open set $\Omega \subset \mathbb{R}^2$.

(b) Let $T$ be a distribution on $\mathbb{R}^2$, and suppose that $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0$ on all of $\mathbb{R}^2$. Prove that there is a constants $A$ so that for every function $\varphi \in C^0_c(\mathbb{R}^2)$, $T(\varphi) = A \int_{\mathbb{R}^2} \varphi(x) \, dx$.

(c) Suppose that $T$ is a distribution on $\mathbb{R}^2$ and suppose that $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0$ on the open set $\mathbb{R}^2 \setminus \{0\}$. Is the conclusion of part (b) still true? Either give a proof, or give a counter-example.

(d) Define a real valued function $g$ on $\mathbb{R}^2$ by setting $g(x, y) = \begin{cases} x^2 + y^2 & \text{if } x^2 + y^2 \leq 1 \\ 1 & \text{if } x^2 + y^2 > 1 \end{cases}$. Show that $\Delta g$ (in the sense of distributions) is a finite measure on $\mathbb{R}^2$, and calculate what that measure is.

Problem IX A real valued function $f$ defined on $\mathbb{R}$ belongs to the space $C^{1/2}(\mathbb{R})$ if and only if

$$\sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\sqrt{|x - y|}} < +\infty.$$ 

Prove that a function $f \in C^{1/2}(\mathbb{R})$ if and only if there is a constant $C$ so that for every $\varepsilon > 0$ there is a bounded function $\varphi \in C^\infty(\mathbb{R})$ such that

$$\sup_{x \in \mathbb{R}} |f(x) - \varphi(x)| \leq C \sqrt{\varepsilon} \quad \text{and} \quad \sup_{x \in \mathbb{R}} \sqrt{\varepsilon} |\varphi'(x)| \leq C.$$
Problem VII  Evaluate the convergent improper Riemann integral
\[ \int_{-\infty}^{+\infty} \frac{\cos \left( \frac{\pi x}{2} \right)}{x^4 - 1} \, dx. \]

Be sure to justify all your calculations.

Problem VII

(a) For a complex number \( z \in \mathbb{C} \), let
\[ f_N(z) = \prod_{n=0}^{N} \left[ (1 - e^{-2\pi n} e^{2\pi i z}) (1 - e^{-2\pi n} e^{-2\pi i z}) \right]. \]
Show the sequence \( \{f_N\} \) converges as \( N \to \infty \) to an entire holomorphic function \( f(z) \) and prove that \( f(z) = 0 \) if and only if \( z = a + ib \) for where \((a, b)\) is a pair of integers.

(b) Does there exist an entire holomorphic function \( g \) such that \( g(z) = g(z + 1) = g(z + i) \) for all complex numbers \( z \) and such that \( g(z) = 0 \) if and only if \( z = a + ib \) where \((a, b)\) is a pair of integers? Why or why not?

(c) Show that the sum
\[ \varphi(z) = \frac{1}{z^2} + \sum_{m+n\neq0} \left( \frac{1}{(z + m + nj)^2} - \frac{1}{(m + nj)^2} \right) \]
converges to a meromorphic function.

(d) Prove that \( \varphi'(z) = \varphi'(z + 1) = \varphi'(z + i) \) for all complex numbers \( z \), and then show that the same identities are true for the function \( \varphi(z) \).

Problem IX  Let \( f \) be a holomorphic function in the unit disk \( \mathbb{D} \).

(a) Suppose that \( |f(z)| \leq M \) for all \( z \in \mathbb{D} \). Prove that \( |f'(z)| \leq M (1 - |z|)^{-1} \) for all \( z \in \mathbb{D} \).

(b) Suppose that \( f(0) = 0 \) and that \( \iint_{\mathbb{D}} |f'(z + iy)|^2 \, dz \, dy = A^2 < +\infty \). Show that for all \( z \in \mathbb{D} \),
\[ |f(z)| \leq A \log(1 - |z|). \]
Problem I

(a) Show that \( n^n e^{-n} \leq n! \leq n^n \) for all positive integers \( n \).

(b) Let \( \{c_n\}, \ n = 1, 2, \ldots, \) be a sequence of positive real numbers. Suppose there is a real number \( \alpha \in \mathbb{R} \) and a constant \( C > 0 \) so that for all \( n \geq 1 \)

\[
\frac{c_{n+1}}{c_n} = 1 + \frac{\alpha}{n} + R(n) \quad \text{where} \quad |R(n)| \leq \frac{C}{n^2}.
\]

Show that, depending on \( \alpha \), the sequence \( \{c_n\} \) has a limit which is either zero, positive, or infinite.

(c) Using the results of part (b), show that the sequence \( \left\{ \frac{n!}{n^n e^{-n} \sqrt{n}} \right\}, \ n = 1, 2, \ldots, \) has a finite non-zero limit.

Problem II  \( \text{(NOTE: This is an advanced calculus problem. Do not quote theorems from the theory of Lebesgue integration for its solution.)} \)

For each positive integer \( n \) let \( a^{(n)} \) be an infinite sequence of complex numbers so that

\[
a^{(n)} = \left( a_0^{(n)}, a_1^{(n)}, \ldots, a_j^{(n)}, \ldots \right).
\]

(a) Suppose there is a positive real number \( M \) such that for all \( n \)

\[
\sum_{j=0}^{+\infty} |a_j^{(n)}|^2 \leq M.
\]

Prove that

\[
\sum_{j=0}^{+\infty} \liminf_{n \to \infty} |a_j^{(n)}|^2 \leq M.
\]

Is it also true that

\[
\sum_{j=0}^{+\infty} \limsup_{n \to \infty} |a_j^{(n)}|^2 \leq M?
\]

Either prove this or show that it is false by giving a counter-example.

(b) Assume that for each \( j, \lim_{n \to \infty} a_j^{(n)} = \alpha_j \) exists. If

\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} |a_j^{(n)}|^2 = \sum_{j=0}^{\infty} |\alpha_j|^2,
\]

prove that

\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} |a_j^{(n)} - \alpha_j|^2 = 0.
\]

Problem III  \( \text{For } x > 0, \text{ let } F(x) = \int_0^x \frac{1 - e^{-xt^2}}{t^2} \, dt. \)

(a) Show that this improper integral converge.

(b) Show that the function \( F \) is differentiable, and find an explicit formula for \( F'(x) \) in terms of elementary functions.

(c) Use the results of part (b) to find an explicit expression for \( F(x) \) in terms of elementary functions.
Problem IV  
Let \( E \subset \mathbb{R} \) be a proper non-empty measurable set, so that \( \emptyset \neq E \neq \mathbb{R} \). Assume that \( E \) is invariant under translation by rational numbers. Explicitly, this means that if \( x \in E \) and if \( r \) is any rational number, then \( x + r \in E \). Show that either \( E \) has Lebesgue measure zero or that \( \mathbb{R} - E \) has Lebesgue measure zero. Give examples to show that both conclusions are possible.

Problem V  
Let \( \{E_j\}_{j \in \mathbb{N}} \) be a countable collection of measurable subsets of \( \mathbb{R}^d \).

(a) Show that the set \( A \) of points \( x \in \mathbb{R}^d \) that belong to all but finitely many of the sets \( E_j \) is measurable.

(b) Show that if \( \lim_{j \to \infty} |E_j| = 0 \), the set \( A \) defined in part (a) has measure zero.

(c) Show that the set \( B \) of points \( x \in \mathbb{R}^d \) that belong to infinitely many of the set \( E_j \) is measurable.

(d) If \( \lim_{j \to \infty} |E_j| = 0 \), must the set \( B \) defined in part (c) have measure zero? Either prove that this is true, or show that it is false by giving a counter-example.

Problem VI  
For \( t \in \mathbb{R} \) let \( g(t) = (1 + |t|)^{-1} \). Fix \( x \in \mathbb{R} \) and, for each non-zero \( h \in \mathbb{R} \) set
\[
G_h(t) = \frac{g(x + h - t) - g(x - t)}{h}.
\]

(a) Prove that each \( G_h \in L^2(\mathbb{R}) \).

(b) Prove that \( \lim_{h \to 0} G_h(t) \) exists for almost every \( t \in \mathbb{R} \).

(c) Prove that if \( G_0 \) is the limit function found in part (b), then
\[
\lim_{h \to 0} \int_{\mathbb{R}} |G_h(t) - G_0(t)|^2 \, dt = 0.
\]

(d) Let \( f \in L^2(\mathbb{R}) \), and define
\[
f \ast g(x) = \int_{\mathbb{R}} f(t) g(x - t) \, dt.
\]
Prove that \( f \ast g \) is a continuously differentiable function on \( \mathbb{R} \). Be sure to justify all your steps including the existence of the integral defining \( f \ast g \), the continuity of this function, and the continuous differentiability of this function.

Problem VII  
Assume that \( 0 < \alpha < 2 \). Evaluate the definite integral \( \int_0^\infty \frac{\log(1 + x^2)}{x^{1+\alpha}} \, dx \). (Hint: First integrate by parts.)
Problem VIII

Let $L^2(\mathbb{D})$ denote the space of square integrable complex valued measurable functions on the open unit disk $\mathbb{D}$ in the complex plane with norm $||f||_2 = \left( \iint_{\mathbb{D}} |f(z)|^2 \, dm(z) \right)^{\frac{1}{2}}$, where $dm(z)$ denotes Lebesgue measure on $\mathbb{D}$. Let

$$B^2(\mathbb{D}) = \left\{ f \in L^2(\mathbb{D}) \mid f \text{ is holomorphic on } \mathbb{D} \right\}.$$

(a) Prove that there is a constant $C$ so that for all $f \in B^2(\mathbb{D})$ and all $z \in \mathbb{D},$

$$|f(z)| \leq \frac{C}{1 - |z|} ||f||_2.$$

(b) Show that equation in part (a) is sharp in the sense that there does not exist a constant $\alpha$ with $\alpha < 1$ so that

$$|f(z)| \leq \frac{C}{(1 - |z|)^{\alpha}} ||f||_2.$$

(c) Let $B_N(z, w) = \frac{1}{\pi} \sum_{j=0}^{N} (j + 1)z^j w^j$, and for $f \in L^2(\mathbb{D})$, set

$$B_N[f](z) = \iint_{\mathbb{D}} B_N(z, w) f(w) \, dm(w).$$

Show that if $f \in L^2(\mathbb{D})$, then the sequence $\{B_N[f]\}$ belongs to $B^2(\mathbb{D})$ and converges both in the norm $|| \cdot ||_2$ and uniformly on compact subsets of $\mathbb{D}$.

(d) If $f \in B^2(\mathbb{D})$, prove that

$$\lim_{N \to \infty} B_N[f] = f.$$

Be sure to indicate in what sense you are considering the limit.

Problem IX

(a) Prove that the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}$$

converges to a meromorphic function on $\mathbb{C}$.

(b) Prove that there is an entire function $f(z)$ so that

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2} + f(z).$$

(c) Prove that the function $f$ found in part (b) is identically zero.

(d) Evaluate $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2}.$
Problem IV  
Let \( E \subset \mathbb{R} \) be a proper non-empty measurable set, so that \( \emptyset \neq E \neq \mathbb{R} \). Assume that \( E \) is invariant under translation by rational numbers. Explicitly, this means that if \( x \in E \) and if \( r \) is any rational number, then \( x + r \in E \). Show that either \( E \) has Lebesgue measure zero or that \( \mathbb{R} - E \) has Lebesgue measure zero. Give examples to show that both conclusions are possible.

Problem V  
Let \( \{E_j\}_{j \in \mathbb{N}} \) be a countable collection of measurable subsets of \( \mathbb{R}^d \).

(a) Show that the set \( A \) of points \( x \in \mathbb{R}^d \) that belong to all but finitely many of the sets \( E_j \) is measurable.

(b) Show that if \( \lim_{j \to \infty} |E_j| = 0 \), the set \( A \) defined in part (a) has measure zero.

(c) Show that the set \( B \) of points \( x \in \mathbb{R}^d \) that belong to infinitely many of the set \( E_j \) is measurable.

(d) If \( \lim_{j \to \infty} |E_j| = 0 \), must the set \( B \) defined in part (c) have measure zero? Either prove that this is true, or show that it is false by giving a counter-example.

Problem VI  
For \( t \in \mathbb{R} \) let \( g(t) = (1 + |t|)^{-1} \). Fix \( x \in \mathbb{R} \) and, for each non-zero \( h \in \mathbb{R} \) set

\[
G_h(t) = \frac{g(x + h - t) - g(x - t)}{h}.
\]

(a) Prove that each \( G_h \in L^2(\mathbb{R}) \).

(b) Prove that \( \lim_{h \to 0} G_h(t) \) exists for almost every \( t \in \mathbb{R} \).

(c) Prove that if \( G_0 \) is the limit function found in part (b), then

\[
\lim_{h \to 0} \int_{\mathbb{R}} |G_h(t) - G_0(t)|^2 \, dt = 0.
\]

(d) Let \( f \in L^2(\mathbb{R}) \), and define

\[
f * g(x) = \int_{\mathbb{R}} f(t) \cdot g(x - t) \, dt.
\]

Prove that \( f * g \) is a continuously differentiable function on \( \mathbb{R} \). Be sure to justify all your steps including the existence of the integral defining \( f * g \), the continuity of this function, and the continuous differentiability of this function.

Problem VII  
Let \( C_0^\infty(\mathbb{R}) \) denote the space of infinitely differentiable complex valued functions with compact support on \( \mathbb{R} \).

(a) Show that for any \( \varphi \in C_0^\infty(\mathbb{R}) \),

\[
\lim_{\epsilon \to 0} \int_{|t| > \epsilon} \frac{\varphi(t)}{t} \, dt
\]

exists. Denote this limit by \( T[\varphi] \).

(b) Show that the linear functional \( T \) defined in part (a) is a distribution on \( \mathbb{R} \). In particular, check carefully that all the hypotheses in the definition of a distribution are satisfied.

(c) If \( S \) is a distribution on \( \mathbb{R} \), define carefully what is meant by the support of \( S \).

(d) Find two distributions \( T_1 \) and \( T_2 \) on \( \mathbb{R} \) whose supports are, respectively, \( (-\infty, 0] \) and \( [0, +\infty) \), such that if \( T \) is the distribution defined in part (a), then \( T = T_1 + T_2 \).
Problem VIII. In this problem \( L^2 \) stands for \( L^2(0, 1) \) and its norm is denoted simply by \( || \cdot || \); also \( W^{1,2} = \{ f \in L^2 \mid f' \in L^2 \} \), with norm \( ||f||_{W} = (||f||^2 + ||f'||^2)^{\frac{1}{2}} \); finally \( C^1 = C^1([0, 1]) \), with norm \( ||f||_1 = \text{Sup} |f| + \text{Sup} |f'| \).

(a) Let \( B \) be the closed unit ball of \( W^{1,2} \); show that it is a compact subset of \( L^2 \).

(b) Let \( B' \) be the closed unit ball of \( C^1 \); show that it is a relatively compact subset of \( L^2 \). Is it a compact subset of \( L^2 \)?

(c) Let \( E \) be a closed subspace of \( L^2 \), such that \( E \cap W^{1,2} = \{0\} \) (for example the linear span of a function in \( L^2 \) and not in \( W^{1,2} \)). Let \( \varphi \) be a continuous linear form on \( E \), continuous with respect to the \( L^2 \) norm. Show that for every \( \epsilon > 0 \), there exists a continuous linear form \( \hat{\varphi} \) on \( L^2 \), whose restriction to \( E \) is \( \varphi \), and such that

\[
\sup_{f \in B} |\hat{\varphi}(f)| \leq \epsilon
\]

with \( B \) as in part (a). Can one take \( \epsilon = 0 \)?

You can use the following geometric form of the Hahn Banach Theorem: If \( K \) is a convex compact subset of a normed vector space \( E \), and \( L \) is a closed affine subspace of \( E \) that does not intersect \( K \), there exists a close affine hyperplane containing \( L \) and still not intersecting \( K \).

Problem IX. Let \( H \) be a separable Hilbert space with norm \( || \cdot ||_H \) and inner product \( \langle \cdot, \cdot \rangle_H \). Let \( \{\varphi_n\} \), \( n = 1, 2, \ldots \), be a complete orthonormal basis for \( H \). Let \( 0 < \delta < 1 \) and let \( \{f_n\} \) be a sequence of elements of \( H \) such that for every finite set of complex numbers \( \{a_n\} \) we have

\[
\left| \sum_{n=1}^{\infty} a_n(\varphi_n - f_n) \right|_H^2 \leq \delta^2 \sum |a_n|^2.
\]

(a) Prove that the series \( K[x] = \sum_{n=1}^{\infty} \langle x, \varphi_n \rangle_H \langle \varphi_n - f_n \rangle \) converges in norm for every \( x \in H \).

(b) Prove that \( K \) defines a bounded linear transformation from \( H \) to \( H \), and show that if \( K^* \) is the adjoint operator, then \( ||K^*|| \leq \delta \).

(c) Prove that for each \( n \geq 1 \), \( (I - K)[\varphi_n] = f_n \), and there exists a unique element \( g_n \in H \) such that \( (I - K^*)[g_n] = \varphi_n \). Hint: Prove that if an operator \( T \) has operator norm less than 1, then \( I - T \) is invertible.

(d) Prove that for \( m, n \geq 1 \), \( \langle f_n, g_m \rangle_H = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases} \)

(e) Prove that for every \( x \in H \), \( x = \sum_{n=1}^{\infty} \langle x, g_n \rangle_H f_n \) where the series converges in the norm in \( H \).
Problem I  Let $f$ be a positive decreasing function defined on $(0, \infty)$. This means that if $0 < a < b < \infty$, then $f(a) \geq f(b) > 0$. Let $\epsilon > 0$ be a fixed positive number.

(a) Suppose that for all $0 < x < \infty$, $f(2x) \leq 2^{-1-\epsilon} f(x)$. Prove that there is a constant $C$ depending only on $\epsilon$ so that for $a > 0$,
\[ \int_a^\infty f(x) \, dx \leq C a f(a). \]

(b) Suppose that for all $0 < x < \infty$, $f(x) \leq 2^{1-\epsilon} f(2x)$. Prove that there is a constant $C$ depending only on $\epsilon$ so that for $a > 0$,
\[ \int_0^a f(x) \, dx \leq C a f(a). \]

(c) Suppose that for all $0 < x < \infty$, $f(2x) \geq 2^{-1} f(x)$. Prove that the improper integral $\int_1^\infty f(x) \, dx$ diverges.

Problem II  For $a, b > 0$, let
\[ F(a, b) = \int_{-\infty}^{+\infty} \frac{dx}{x^4 + (x - a)^4 + (x - b)^4}. \]

For which $p > 0$ is it true that
\[ \int_0^1 \int_0^1 F(a, b)^p \, da \, db < +\infty? \]

HINT: Do not try to evaluate the integral defining $F(a, b)$ directly. Instead, first suppose $a \leq b$ and show that there are positive constants $C_1$ and $C_2$ so that
\[ C_1 \leq b^3 F(a, b) \leq C_2. \]

Problem III  Let $\{z_1, z_2, \ldots, z_n, \ldots\}$ be a sequence of complex numbers, and suppose that
\[ \lim_{n \to \infty} z_n = L \]
exists. Prove that
\[ \lim_{n \to \infty} \frac{1}{n^2} \left( z_1 + 3z_2 + 5z_3 + \cdots + (2n-1)z_n \right) = L. \]

Problem IV  Let $\{E_n\}$, $n = 2, 3, 4, \ldots$ be a sequence of open subsets of $\mathbb{R}$ defined as follows:
\[ E_2 = (0, 1) \]
\[ E_3 = \left( 0, \frac{1}{3} \right) \cup \left( \frac{2}{3}, 1 \right) \]
\[ E_4 = \left( 0, \frac{3}{24} \right) \cup \left( \frac{5}{24}, \frac{1}{3} \right) \cup \left( \frac{2}{3}, \frac{19}{24} \right) \cup \left( \frac{21}{24}, 1 \right) \], etc.

where $E_n$ is the union of open intervals, all of the same length $l_n$, and $E_{n+1}$ is obtained from $E_n$ by removing a closed interval of length $\frac{l_n}{n+1}$ from the center of each interval of $E_n$. Set $E = \bigcap_{n=2}^{\infty} E_n$.

(a) Prove that $E$ is an uncountable Borel set, and find the Lebesgue measure of $E$.

(b) The set $E_n$ is the disjoint union of $k_n$ intervals, each of length $l_n$. For $\alpha \in [0, 1]$ find $\lim_{n \to \infty} k_n l_n^\alpha$.

(c) What does the result in part (b) indicate about the Hausdorff dimension of the set $E$? (You are not asked for a complete proof, which may be somewhat technical. You are only asked to show understanding of the definitions of Hausdorff dimension and Hausdorff measure.)
Problem V  Let \( \{f_n\} \) be a sequence of real-valued continuous functions on \( \mathbb{R}^3 \). Suppose that the sequence \( \{f_n\} \) converges pointwise to a function \( f \). Assume that each function \( f_n \) and the limit function \( f \) are integrable. Assume that there are real numbers \( m \) and \( p \) so that for all \( x \in \mathbb{R}^3 \)

\[-\frac{1}{1 + |x|^m} \leq f_n(x) \leq \frac{1}{1 + |x|^p}.

(a) Consider the statement— "If for each \( n \), \( \iiint_{\mathbb{R}^3} f_n(x) \, dx = 0 \), then \( \iiint_{\mathbb{R}^3} f(x) \, dx = 0. \)" For which \( m \) and \( p \) is this statement true.

(b) Consider the statement— "If for each \( n \), \( \iiint_{\mathbb{R}^3} f_n(x) \, dx \geq 0 \), then \( \iiint_{\mathbb{R}^3} f(x) \, dx \geq 0. \)" For which \( m \) and \( p \) is this statement true.

*Be sure to give proofs of the positive statements you make in parts (a) and (b). You only need to give counter-examples for one of the two parts.*

Problem VI  Let \( f \) be a measurable, integrable function on \( \mathbb{R}^2 \).

(a) Show that you can approximate \( f \) be a sequence of continuous functions \( \{f_n\} \) in such a way that for almost every \( y \), \( \int_{\mathbb{R}} |f_n(x, y) - f(x, y)| \, dx \to 0. \)

Now set \( \hat{f}(x, y) = \int_{x-1}^{x+1} f(t, y) \, dt. \)

(b) Show that \( \hat{f} \) is defined almost everywhere.

(c) Use the result of part (a) to show that the function \( \hat{f} \) is measurable.

*In doing this problem you may use without proof the following version of Fubini’s theorem: If \( f \) is a measurable, integrable function on \( \mathbb{R}^2 \), set \( f_x(y) = f(x, y) = f^y(x) \). Then:

1. For almost all \( x \in \mathbb{R} \), \( f_x \in L^1(\mathbb{R}) \), and for almost all \( y \in \mathbb{R} \), \( f^y \in L^1(\mathbb{R}) \).

2. The functions \( g(x) = \int_{\mathbb{R}} f_x(y) \, dy \) and \( h(y) = \int_{\mathbb{R}} f^y(x) \, dx \) are defined almost everywhere, and belong to \( L^1(\mathbb{R}) \).

3. \( \iint_{\mathbb{R}^2} f(x, y) \, dA = \int_{\mathbb{R}} g(x) \, dx = \int_{\mathbb{R}} h(y) \, dy. \)

Problem VII  If \( f \) is continuous with compact support in \((0, \infty)\) set

\[ H[f](x) = \int_0^\infty \frac{f(y)}{x + y} \, dy. \]

(a) Prove that there does \textit{not} exist a constant \( C \) such that for every \( f \) continuous with compact support in \((0, \infty)\) we have

\[ ||H[f]||_{L^\infty(0, \infty)} \leq C ||f||_{L^\infty(0, \infty)}. \]

(b) Prove that there does \textit{not} exist a constant \( C \) such that for every \( f \) continuous with compact support in \((0, \infty)\) we have

\[ ||H[f]||_{L^1(0, \infty)} \leq C ||f||_{L^1(0, \infty)}. \]

\textit{HINT: Consider the adjoint of the operator \( H \) and use duality.}

(c) Prove that for every \( 1 < p < \infty \) there is a constant \( C_p \) such that

\[ ||H[f]||_{L^p(0, \infty)} \leq C_p ||f||_{L^p(0, \infty)}. \]

\textit{HINT: In the integral defining \( H[f](x) \), make the change of variables \( y = x/s \) and apply Minkowski’s inequality for integrals.}

(d) Prove that \[ \left\{ x > 0 \mid |H[f](x)| > \lambda \right\} \leq \lambda^{-1} ||f||_{L^1(0, \infty)}. \]
Problem VIII  Let $F$ be a real-valued continuous function with compact support defined on the real line $\mathbb{R}$. We say that $F$ satisfies a Hölder condition of order $\alpha$ for some $0 < \alpha < 1$ if there is a constant $C$ so that for all $a, b \in \mathbb{R}$ we have

$$|F(b) - F(a)| \leq C |b - a|^{\alpha}$$

(a) Suppose that $F$ is a continuous function with compact support on $\mathbb{R}$, and suppose for every $\epsilon > 0$ there is a continuously differentiable function $G$ such that for all $x \in \mathbb{R}$,

$$|F(x) - G(x)| \leq \epsilon^\alpha$$

$$|G'(x)| \leq \epsilon^{\alpha-1}.$$

Prove that $F$ satisfies a Hölder condition of order $\alpha$.

(b) Suppose that $F$ is a continuous function with compact support on $\mathbb{R}$ and that $F$ satisfies a Hölder condition of order $\alpha$ with $0 < \alpha < 1$. Prove that there is an infinitely differentiable function $G$ with compact support on $\mathbb{R}$ and a constant $C$ so that for all $x \in \mathbb{R}$,

$$|F(x) - G(x)| \leq C \epsilon^\alpha$$

$$|G'(x)| \leq C \epsilon^{\alpha-1}.$$

Problem IX  As usual, let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denote the Laplace operator on $\mathbb{R}^2$. Also, set

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}.$$

(a) What is meant by the distribution on $\mathbb{R}^2$ defined by the function $f$, and what is the definition of the distribution $\Delta f$?

(b) Let $\varphi \in C_\infty^0(\mathbb{R}^2)$ and assume that $\varphi(0,0) = \frac{\partial \varphi}{\partial x}(0,0) = \frac{\partial \varphi}{\partial y}(0,0) = 0$. Prove that $\frac{\varphi(x,y)}{(x^2 + y^2)^{\frac{3}{2}}}$ is an integrable function on $\mathbb{R}^2$, and then prove that

$$\langle \Delta f, \varphi \rangle = \iint_{\mathbb{R}^2} \frac{\varphi(x,y)}{(x^2 + y^2)^{\frac{3}{2}}} \, dx \, dy.$$
Problem V  Let \( \{f_n\} \) be a sequence of real-valued continuous functions on \( \mathbb{R}^3 \). Suppose that the sequence \( \{f_n\} \) converges pointwise to a function \( f \). Assume that each function \( f_n \) and the limit function \( f \) are integrable. Assume that there are real numbers \( m \) and \( p \) so that for all \( x \in \mathbb{R}^3 \)
\[
\frac{1}{1+|x|^m} \leq f_n(x) \leq \frac{1}{1+|x|^p}.
\]
(a) Consider the statement—"If for each \( n \), \( \int_{\mathbb{R}^3} f_n(x) \, dx = 0 \), then \( \int_{\mathbb{R}^3} f(x) \, dx = 0 \)." For which \( m \) and \( p \) is this statement true.
(b) Consider the statement—"If for each \( n \), \( \int_{\mathbb{R}^3} f_n(x) \, dx \geq 0 \), then \( \int_{\mathbb{R}^3} f(x) \, dx \geq 0 \)." For which \( m \) and \( p \) is this statement true.

Be sure to give proofs of the positive statements you make in parts (a) and (b). You only need to give counter-examples for one of the two parts.

Problem VI  Let \( f \) be a measurable, integrable function on \( \mathbb{R}^2 \).

(a) Show that you can approximate \( f \) by a sequence of continuous functions \( \{f_n\} \) in such a way that for almost every \( y \),
\[
\int_{\mathbb{R}} |f_n(x,y) - f(x,y)| \, dx \to 0.
\]

Now set \( \tilde{f}(x,y) = \int_{-1}^{x+1} f(t,y) \, dt \).

(b) Show that \( \tilde{f} \) is defined almost everywhere.

(c) Use the result of part (a) to show that the function \( \tilde{f} \) is measurable.

In doing this problem you may use without proof the following version of Fubini's theorem: If \( f \) is a measurable, integrable function on \( \mathbb{R}^2 \), set \( f_z(y) = f(x,y) = f^y(x) \). Then:

1. For almost all \( x \in \mathbb{R} \), \( f_z \in L^1(\mathbb{R}) \), and for almost all \( y \in \mathbb{R} \), \( f^y \in L^1(\mathbb{R}) \).
2. The functions \( g(x) = \int_{\mathbb{R}} f_z(y) \, dy \) and \( h(y) = \int_{\mathbb{R}} f^y(x) \, dx \) are defined almost everywhere, and belong to \( L^1(\mathbb{R}) \).
3. \[
\int_{\mathbb{R}^2} f(x,y) \, dA = \int_{\mathbb{R}} g(x) \, dx = \int_{\mathbb{R}} h(y) \, dy.
\]

Problem VII  Let
\[
\Omega = \left\{ r e^{i\theta} \in \mathbb{C} \mid 0 < r < 1, \quad and \quad -\pi < \theta < +\pi \right\} \subset \mathbb{D}.
\]

Assume that \( f : \Omega \to \mathbb{D} \) is holomorphic, one-to-one, and onto. Show that there is no continuous function \( g : \mathbb{D} \to \mathbb{C} \) such that \( f(z) = g(z) \) for all \( z \in \Omega \).

Problem VIII  Let \( \mathbb{D}^* = \mathbb{D} \setminus \{0\} \) denote the punctured unit disk. Suppose that \( f : \mathbb{D}^* \to \mathbb{D}^* \) is holomorphic, and has the property that the equation \( f(z) = w \) has exactly two solutions \( z \in \mathbb{D}^* \) for every \( w \in \mathbb{D}^* \), counting multiplicity. Prove that there is a complex number \( \lambda \) with \( |\lambda| = 1 \) so that \( f(z) = \lambda z^2 \).

HINT: Consider the mapping which takes \( z_0 \in \mathbb{D} \) to the second root of the equation \( f(z) = f(z_0) \).

Problem IX  
(a) Show that on the complex plane slit along the positive real axis from 1 to \( +\infty \) and along the negative real axis from \(-1 \) to \( -\infty \), there is a holomorphic function \( h \) satisfying \( h(z)^2 = (z^2-1)^{-1} \) such that \( h(0) = -i \). Compute: \( \lim_{y \to 0^+} h(2+i y) \), \( \lim_{y \to 0^-} h(2-i y) \), \( \lim_{y \to 0^+} h(-2+i y) \), and \( \lim_{y \to 0^-} h(-2-i y) \).

(b) Evaluate \( \int_1^\infty \frac{(x^2-1)^{-\frac{1}{2}} \, dx}{x} \). (You may use any method you choose.)
1. (i) Let \( \{f_n\} \) be a sequence of \( C^1 \) functions on a compact interval \( I \) such that 
\[ |f_n(x)| + |f'_n(x)| \leq M \text{ for all } x \in I \text{ and } n = 1, 2, 3, \ldots \] 
Show that there is a subsequence \( \{f_{n_k}\} \) which converges uniformly on \( I \).

(ii) Is the preceding statement still true if we drop the assumption that \( I \) is compact? (Proof or counterexample)

(iii) Can one also show that under the assumptions in (i) the sequence \( f_n \) has a subsequence whose derivatives converge uniformly? (Proof or counterexample)

2. Let \( \{a_n\}_{n=1}^\infty \) be a numerical sequence and let 
\[ b_n = \frac{1}{n^5} \sum_{k=1}^n k^5 a_k. \]

(i) Prove or disprove: If \( a_n \) converges then \( b_n \) converges.
(ii) Prove or disprove: If \( b_n \) converges then \( a_n \) converges.

*Hint:* Relate \( \sum_{k=1}^n k^5 \) to an integral.

3. (i) Suppose that \( \mathcal{O} \subset \mathbb{R}^n \) is open, \( f: \mathcal{O} \to \mathbb{R} \) is a \( C^\infty \) function and \( n > 1 \). Show that \( f \) is not a one to one function.

(ii) Suppose that \( \mathcal{O} \subset \mathbb{R}^n \) is open, \( f: \mathcal{O} \to \mathbb{R}^k \) is a \( C^\infty \) function and \( n > k \). Show that \( f \) is not a one to one function.

*Hint:* Use induction on \( k \).

4. Suppose that the sequence \( \{f_n\} \) of nonnegative Lebesgue measurable functions on \( \mathbb{R} \) converges to \( f \) pointwise, and suppose that \( \int_{\mathbb{R}} f_n(x)dx < \infty \) for all \( n \), \( \int_{\mathbb{R}} f(x)dx < \infty \) and \( \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx \).

Prove that for all measurable sets \( E \) we have
\[ \lim_{n \to \infty} \int_{E} f_n(x)dx = \int_{E} f(x)dx \]

5. (i) Let \( f: \mathbb{R} \to \mathbb{C} \) be continuous on \([0, 1]\) and assume that \( f \) is 1-periodic, i.e. \( f(x+1) = f(x) \) for all \( x \). Let \( \beta \) be an irrational number in \((0, 1)\) and define
\[ \mathcal{G}_n f(x) = \frac{1}{n} \sum_{k=1}^n f(x + k\beta) \]

Show that for all \( x \in \mathbb{R} \)
\[ \lim_{n \to \infty} \mathcal{G}_n f(x) = \int_0^1 f(t)dt \]

and that the convergence is uniform on \( \mathbb{R} \).

*Hint:* Show first that the formula for the limit is correct for \( f_m(x) = e^{2\pi imx} \), \( m \in \mathbb{Z} \).

(ii) Formulate and prove a generalization for \( f \in L^p \), for \( 1 \leq p < \infty \).
6. (i) Show that for nonnegative scalars \(a, b \in \mathbb{R}\) and \(p \geq 2\) we have
\[
a^p + b^p \leq (a^2 + b^2)^{p/2}
\]
and
\[
\left(\frac{a^2 + b^2}{2}\right)^{p/2} \leq \frac{a^p}{2} + \frac{b^p}{2}.
\]

*Hint:* For the second inequality use the convexity of \(t \mapsto t^{p/2}\) for \(t > 0\).

(ii) Show that for \(f, g \in L^p(X, d\mu)\) and \(2 \leq p < \infty\)
\[
\frac{\|f + g\|_p^p}{2} + \frac{\|f - g\|_p^p}{2} \leq \frac{\|f\|_p^p}{2} + \frac{\|g\|_p^p}{2}.
\]

(iii) Show that each closed convex set in \(L^p(2 \leq p < \infty)\) has an element \(f\) of minimal norm.

7. Prove that there is a unique \(C^\infty\) function \(f\) defined on \([0, 1]\) which satisfies the integral equation
\[
f(x) + \int_0^1 \frac{t \cos(tx) f(t)}{1 + f(t)^2} dt = 1
\]
for all \(x \in [0, 1]\).

8. State and prove Baire's theorem, and describe a concrete application.

9. Let \(E\) be a Banach space and let \(E'\) its dual. For a sequence \(\{x_n\}\) in \(E\) we say that \(x_n\) converges weakly to \(x\) if \(\lambda(x_n) \to \lambda(x)\) for all \(\lambda \in E'\).

   (i) Show that if \(x_n\) converges weakly to \(x\) then \(\|x_n\|\) is a bounded sequence and
   \[\|x\| \leq \liminf_{n \to \infty} \|x_n\|\].

   (ii) Show that \(f_n(x) = \chi_{[n, n+1]}(x)\) defines a sequence which converges weakly in the space \(L^p(\mathbb{R})\) if \(1 < p < \infty\).

   (iii) Let \(\{a_n\}\) be a numerical sequence with \(\lim_{n \to \infty} a_n = \infty\), and let \(g_n(x) = a_n \chi_{[n, n+1]}(x)\). Does the sequence \(g_n\) converge weakly in \(L^p\)?
6. (i) Show that for nonnegative scalars \( a, b \in \mathbb{R} \) and \( p \geq 2 \) we have

\[
a^p + b^p \leq (a^2 + b^2)^{p/2}
\]

and

\[
\left(\frac{a^2 + b^2}{2}\right)^{p/2} \leq \frac{a^p}{2} + \frac{b^p}{2}.
\]

*Hint:* For the second inequality use the convexity of \( t \mapsto t^{p/2} \) for \( t > 0 \).

(ii) Show that for \( f, g \in L^p(X, d\mu) \) and \( 2 \leq p < \infty \)

\[
\left\| \frac{f + g}{2} \right\|_p^p + \left\| \frac{f - g}{2} \right\|_p^p \leq \frac{\|f\|_p^p}{2} + \frac{\|g\|_p^p}{2}.
\]

(iii) Show that each closed convex set in \( L^p \) \((2 \leq p < \infty)\) has an element \( f \) of minimal norm.

7. Suppose \( u \) is real-valued and harmonic in \( \mathbb{C} \) and suppose that for all \( r > 0 \)

\[
\max_{0 \leq \theta \leq 2\pi} u(re^{i\theta}) \leq M(r),
\]

with \( 0 \leq M(r) < \infty \).

a) Show that

\[
\frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|d\theta + u(0) \leq 2M(r).
\]

b) Suppose that \( f \) is an entire function and there are non-negative constants \( A, B \) and \( \lambda \) so that

\[
\text{Re} f(re^{i\theta}) \leq Ar^\lambda + B
\]

for all \( r \) and \( \theta \). Show that \( f \) is a polynomial of degree at most \( \lambda \).

8. Evaluate

\[
\int_0^\infty \frac{(\ln x)^2}{1 + x^2} dx.
\]

9. Fix \( 0 < r < R \) and let \( A = \{z \in \mathbb{C} : r < |z| < R\} \). Suppose that \( f \) is holomorphic and non vanishing on \( A \) and continuous on the closure of \( A \) and that \( |f(re^{i\theta})| \equiv \alpha \) and \( |f(Re^{i\theta})| \equiv \beta \) for some constants \( \alpha \) and \( \beta \) and for all \( 0 \leq \theta \leq 2\pi \).

Show that \( f(z) = cz^n \) for some \( c \in \mathbb{C} \) and some \( n = 0, \pm 1, \pm 2, \ldots \).

*Hint:* Consider the function \( u(z) = \log |f(z)| - \gamma \log |z| \) for an appropriate value of \( \gamma \).
1. (i) Let \( \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \) be sequences of real numbers such that

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 1.
\]

(a) Prove: If \( a_n > 0 \) for all \( n \) then \( \sum_{n=1}^{\infty} a_n \) converges if and only if \( \sum_{n=1}^{\infty} b_n \) converges.

(b) Is the statement in (a) still correct if one drops the assumption of positivity of the \( a_n \)? Give a proof or a counterexample.

(ii) Suppose that \( \sum a_n \) converges and \( \sum_{n=1}^{\infty} a_n = A \). Let \( \rho_n = \sum_{k=1}^{n} (1 - \frac{k}{n})a_k \). Prove that \( \lim_{n \to \infty} \rho_n = A \).

2. Let \( a \in \mathbb{R} \) and define \( f_a : \mathbb{R}^n \to \mathbb{R} \) by

\[
f_a(x) = (x_1^4 + x_2^4 + \cdots + x_{n-1}^4 + x_n^2)^a
\]

(i) Let \( B = \{ x \in \mathbb{R}^n : |x| \leq 1 \} \). For every \( a \in \mathbb{R} \) determine whether

\[
\int_B f_a(x) dx < \infty.
\]

(ii) Let \( S_e \) be the unit sphere centered at \( e = (0, \ldots, 0, 1) \) and let \( d\sigma \) be surface measure on \( S_e \).

For every \( a \in \mathbb{R} \) determine whether

\[
\int_{S_e} f_a d\sigma < \infty.
\]

3. Let \( p > 0 \). Suppose that \( |f|^p \) is integrable on every compact interval in \( \mathbb{R} \) and \( f(x) \) satisfies the equation

\[
f(x) = \int_0^x \cos(xt) \sin(f(t)) |f(t)|^p dt
\]

for all \( x \in \mathbb{R} \). Show that \( f(x) = 0 \) for all \( x \in \mathbb{R} \).
4. For \( f \in L^\infty(\mathbb{R}) \) and \( t > 0 \) define
\[
K(t, f) = \inf_{f = g + h} \{ \|g\|_{\infty} + t\|h'\|_{\infty} \}
\]
so that the infimum is taken over all possible decompositions of \( f = g + h \) where \( g \in L^\infty \) and \( h \) is a \( C^1 \) function with bounded derivative. Moreover let
\[
\omega(t, f) = \sup_{x} \sup_{|s| \leq t} |f(x + s) - f(x)|.
\]
Prove that there exists two positive constants \( C_1, C_2 \) (independent of \( f \) and \( t \)) so that for all \( f \in L^\infty, t > 0 \)
\[
C_1 K(t, f) \leq \omega(t, f) \leq C_2 K(t, f)
\]
(i.e. \( K(t, \cdot) \) and \( \omega(t, \cdot) \) are uniformly equivalent seminorms).

*Hint:* For one inequality one has to efficiently decompose \( f \); try a convolution \( h = \phi_t * f \) for suitable \( \phi_t \).

5. (i) Prove that for \( 1 < p < \infty, \alpha > 1/p \)
\[
(*) \quad \int_{0}^{\infty} x^{-\alpha p} \left( \int_{0}^{x} f(t)dt \right)^{p} dx \leq C_{p} \int_{0}^{\infty} |f(x)x^{1-\alpha}|^{p} dx.
\]
(ii) Is there a \( p \in (1, \infty) \) so that \( (*) \) remains true for \( \alpha = 1/p \)?

6. Let \( f \) be a measurable function on \( I = [0, 1] \) and assume that \( f \notin L^\infty(I) \).
   a) Prove that
   \[
   \lim_{p \to \infty} \|f\|_{p} = \infty.
   \]
   b) Can \( \|f\|_{p} \) tend to \( \infty \) arbitrarily slowly? The precise question is: Is it true that to every positive function \( \Phi \) on \( (0, \infty) \) with \( \Phi(p) \to \infty \) there is a measurable \( f \) with \( \|f\|_{p} \to \infty \) but \( \|f\|_{p} \leq \Phi(p) \) for sufficiently large \( p \)?
7R.
(i) State the Arzela-Ascoli theorem (which deals with compact subsets of \( C(K) \) for compact \( K \)).
(ii) Prove using (i):
Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of 1-periodic\(^1\) functions and assume that
\[
\int_0^1 |f_n(t)|^2 dt \leq 1.
\]
Assume that the derivative \( f_n' \) in the sense of distributions belongs to \( L^2[0, 1] \) and
\[
\int_0^1 |f_n'(t)|^2 dt \leq 1
\]
for all \( n \). Show that there is a subsequence of \( \{f_n\} \) which converges uniformly on \( \mathbb{R} \).

8R. State and prove Baire’s theorem, and describe a concrete application.

9R. Let \( u : \mathbb{R}^3 \to \mathbb{R} \) be defined by
\[
u(x, y, z) = \begin{cases} 
  x & \text{if } x^2 + y^2 + z^2 \leq 1 \text{ and } z > 0 \\
  0 & \text{if } x^2 + y^2 + z^2 > 1 \text{ or } z \leq 0
\end{cases}
\]
(i) Compute the derivative \( \partial u / \partial y \) in the sense of distributions and show that it can be identified with a bounded Borel measure \( \mu \).
(ii) Compute \( \mu(E) \) for \( E = \{ (x, y, z) : 0 < x < y < 1, -\infty < z < 1/2 \} \).

---
\(^1\)Here \( f \) is said to be 1-periodic if \( f(x + 1) = f(x) \) for almost every \( x \in \mathbb{R} \).
7C. (i) State the Arzela-Ascoli theorem (which deals with compact subsets of \( C(K) \) for compact \( K \)).

(ii) Let \( \{f_n\} \) be a sequence of holomorphic functions in \( \{z : |z| < R\} \) and assume that for all \( n = 0, 1, 2, \ldots \)

\[
|f_n(z)| \leq C(|z|)
\]

where \( C(r) < \infty \) for every \( r < R \).

Prove using part (i) that there is a subsequence of \( \{f_n\} \) which converges uniformly on every compact subset of \( \{z : |z| < R\} \).

8C. Let \( \Omega = \{z : 0 < |z| < 1\} \) and let \( f \) be holomorphic in \( \Omega \) so that

\[
\iint_{\Omega} |f(x + iy)|^2 dx \, dy < \infty.
\]

Show that \( f \) has a removable singularity at 0, i.e. can be extended to a holomorphic function on \( \{z : |z| < 1\} \).

9C. (i) Compute

\[
\int_0^\infty e^{-x^2} \, dx
\]

(by a method of your choice).

(ii) Show that the improper integrals

\[
\int_0^\infty e^{-(a+i\lambda)x^2} \, dx
\]

exist for \( \lambda \neq 0 \) and \( a \geq 0 \) and use contour integrals to deduce their values from (i).

(iii) Show that

\[
\lim_{a \to 0^+} \int_0^\infty x^2 e^{-(a+i\lambda)x^2} \, dx
\]

exists for \( \lambda \neq 0 \) and compute this limit.

*Be careful to justify all steps.*
Problem I  Does the series
\[
\sum_{k=1}^{\infty} \frac{\sin \sqrt{k}}{k}
\]
converge?

[Hint: compare \(\sum_{k=M}^{N} \frac{\sin \sqrt{k}}{k}\) with a similar integral.]

Problem II  Let \(E \subset \mathbb{Q}\) be the set of \(x\) whose decimal expansion is of the form \(x = 0.d_1d_2 \cdots d_N\) for some \(N \in \mathbb{N}\), and where \(d_1, \ldots, d_N \in \{1, 2, 3, 4, 5, 6, 7, 8\}\) (so \(d_i \neq 0\) and \(d_i \neq 9\) for \(i = 1, \ldots, N\)). Show that any compact subset of \(E\) is finite.
Can we drop the hypotheses that \(d_i \neq 0\) and \(d_i \neq 9\)?

Problem III  Let \(Q = [0, 1] \times \cdots \times [0, 1] \subset \mathbb{R}^n\) be the unit cube, and consider the function
\[
f(x_1, \ldots, x_n) = \frac{x_1 x_2 \cdots x_n}{x_1^{a_1} + \cdots + x_n^{a_n}},
\]
where the \(a_j\) are positive constants. For which \(a_1 > 0, \ldots, a_n > 0\) is the integral \(\int_Q f(x)dx\) finite?

Problem IV  Let \(1 \leq p < \infty\), and let \(f_n \in L^p(\mathbb{R})\) be a sequence of functions. Suppose
\[
\sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_{L^p} < \infty.
\]
Show that the sequence \(f_n\) converges pointwise almost everywhere.

Problem V  Define
\[
\Lambda(x) = \int_0^\infty \frac{e^{-t}}{\log(1 + e^{-t})} dt.
\]

1. For which \(x \in \mathbb{R}\) is the integrand a Lebesgue integrable function?
2. Show that \(\Lambda(x)\) is a continuous function for \(x \in \mathbb{R}_+\).
3. Show that \(\Lambda(x)\) is differentiable for \(x \in \mathbb{R}_+\), and that the derivative is given by \(\Lambda'(x) = \int_0^\infty e^{-t} t^{x-1} dt\).
Give complete proofs.
Problem VI  Let \( K : \mathbb{R}_+ \to \mathbb{R} \) be a nonnegative measurable function for which
\[
\int_0^\infty \frac{K(t)}{\sqrt{t}} \, dt = A < \infty.
\]
In this problem \( L^2(0, \infty) = \{ f : (0, \infty) \to \mathbb{R} : f \) is measurable and \( \int_0^\infty f(x)^2 \, dx < \infty \}, \) with the usual convention that functions which differ only on a set of measure zero are identified.

1. Show that for any two functions \( f, g \in L^2(0, \infty) \) one has
\[
\int_0^\infty \int_0^\infty K(xy)f(x)g(y) \, dx \, dy \leq A \| f \|_{L^2} \| g \|_{L^2}.
\]
[Hint: try the substitution \( x = z/y; \) or you could try to solve (b) first...]

2. Prove that for any \( f \in L^2(0, \infty) \) the integral
\[
Tf(x) \overset{\text{def}}{=} \int_0^\infty K(xy)f(y) \, dy
\]
converges for almost every \( x \in \mathbb{R}_+ \), and that \( T \) defines a bounded operator on \( L^2(0, \infty) \) (i.e. there is a finite constant \( C \) such that \( \| Tf \|_{L^2} \leq C \| f \|_{L^2} \) for all \( f \in L^2(0, \infty) \)).

Problem VII
Let \( X \) and \( Y \) be Banach spaces, and let \( T_{jk} \), \( j, k \in \mathbb{N} \) be a family of bounded operators from \( X \) to \( Y \). Suppose that for every \( k \in \mathbb{N} \) there exists an \( x \in X \) such that \( \sup_{j \in \mathbb{N}} \| T_{jk} x \|_Y = \infty \). Then prove that an \( x \in X \) exists such that \( \sup_{j \in \mathbb{N}} \| T_{jk} x \|_Y = \infty \) holds for all \( k \in \mathbb{N} \).

Problem VIII
Let \( 1 \leq p \leq \infty \). Consider the operators \( T_\epsilon : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \) given by
\[
T_\epsilon f(x) = \frac{1}{2} \int_{-\epsilon}^{\epsilon} f(x + \epsilon t) \, dt.
\]

1. Show that \( |T_\epsilon f(x) - T_\epsilon f(y)| \leq C|x-y|^{1-1/p} \) for all \( x, y \in \mathbb{R} \) and for some finite constant \( C \), depending on \( f \) and \( \epsilon \) but independent of \( x \) and \( y \). Give \( C \) explicitly in terms of \( f \) and \( \epsilon \).

2. It is known that \( \lim_{\epsilon \to 0} \| T_\epsilon f - f \|_{L^p(\mathbb{R})} = 0 \) for each \( f \in L^p(\mathbb{R}) \). Is it true that
\[
\lim_{\epsilon \to 0} \| T_\epsilon - I_{L^p} \|_{\mathcal{L}(L^p)} = 0,
\]
where \( \| \cdot \|_{\mathcal{L}(L^p)} \) is the operator norm on the space \( \mathcal{L}(L^p) \) of bounded operators on \( L^p(\mathbb{R}) \), and \( I_{L^p} \) is the identity operator on \( L^p \)?

Problem IX
Let \( E = \{ (x, y) \in \mathbb{R}^2 : y \geq |x| \} \), let \( f : \mathbb{R}^2 \to \mathbb{R} \) be the characteristic function of \( E \), and let \( \delta \) be the Dirac measure at 0. Show that in the sense of distributions one has
\[
\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2} = 2\delta.
\]
Problem VI

Let $K : \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative measurable function for which

$$\int_0^\infty \frac{K(t)}{\sqrt{t}} \, dt = A < \infty.$$  

In this problem $L^2(0, \infty) = \{ f : (0, \infty) \to \mathbb{R} : f \text{ is measurable and } \int_0^\infty f(x)^2 \, dx < \infty \}$, with the usual convention that functions which differ only on a set of measure zero are identified.

1. Show that for any two functions $f, g \in L^2(0, \infty)$ one has

$$\int_0^\infty \int_0^\infty K(xy)f(x)g(y) \, dx \, dy \leq A \| f \|_{L^2} \| g \|_{L^2}.$$  

[Hint: try the substitution $x = z/y$; or you could try to solve (b) first...]

2. Prove that for any $f \in L^2(0, \infty)$ the integral

$$Tf(x) \overset{\text{def}}{=} \int_0^\infty K(xy)f(y) \, dy$$

converges for almost every $x \in \mathbb{R}_+$, and that $T$ defines a bounded operator on $L^2(0, \infty)$ (i.e. there is a finite constant $C$ such that $\| Tf \|_{L^2} \leq C \| f \|_{L^2}$ for all $f \in L^2(0, \infty)$.)

Problem VII

Let $D$ be the open unit disk in $\mathbb{C}$.

1. Let $(h_n)$ be a sequence of holomorphic maps from $D$ into $D$. Assume that $|h_n'(0)|$ tends to $1$ as $n$ tends to $\infty$. Show that $h_n(0)$ tends to $0$.

2. Let $F$ be the set of holomorphic maps from $D$ into $(D - \{ \frac{1}{2} \})$. Show that there is a constant $M < 1$ such that for every $f \in F$, $|f'(0)| \leq M$.

The second question can be treated, assuming the result of the first question.

Problem VIII

Let $0 < \epsilon < \infty$, $0 < R < \infty$ and let $D_1, D_2$ be two closed disjoint disks in $\mathbb{C}$. Show that there is an entire function $f$ so that $f(D_1)$ is contained in $\{ z : |z| < \epsilon \}$ and $f(D_2)$ contains $\{ z : |z| < R \}$.

[Hint: Use Runge's theorem.]

Problem IX

Use complex methods to find the value of

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$
Problem I. 

(1) For which $x \in \mathbb{R}$ does the Taylor series of $f(x) = \log(1 - x)$ centered at 0 converge to $f(x)$? 

(2) For which $a > -1$ does the Taylor series converge uniformly on the interval $[-1, a]$? 

(3) Evaluate the sum 

$$
\sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)} \text{ where } |x| \leq 1.
$$

Include all the details of your derivation.

Problem II. Let $a > 0$ and $b > 0$. Prove that there is a unique differentiable function $f$ defined on $(-\infty, \infty)$ satisfying $f(0) = 0$ and 

$$
f'(x) = a - b|f(x)|^{3/2}
$$

for all $x$. Also show that $\lim_{x \to \infty} f(x)$ exists and determine this limit.

Problem III. Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$. 

(1) Suppose the second derivative of $f$ exists at $x_0$ (but not necessarily anywhere else). Show that 

$$
\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} = f''(x_0).
$$

(2) Suppose that 

$$
\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}
$$

exists. Is it true that the second derivative of $f$ exists at $x_0$? Give a proof or a counterexample!
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Problem IV.

(1) Let $E, F \subset \mathbb{R}^N$ be Lebesgue measurable subsets both of which have finite and positive measure. For $x \in \mathbb{R}^N$ we define the translates $E_x = \{x + y : y \in E\}$. Show that the set

$$G = \{x \in \mathbb{R}^N : E_x \cap F \neq \emptyset\}$$

has positive Lebesgue measure. [Hint: Several solutions are possible: you could consider the convolution of suitable characteristic functions; or you could apply Lebesgue’s differentiation theorem to a characteristic function.]

(2) If $E \subset \mathbb{R}$ is an open and dense subset of the real line, must its Lebesgue measure be infinite? (Prove, or give a counterexample.)

Problem V. For $f \in L^1_{loc}(\mathbb{R})$, $j \geq 0$ define the so-called “conditional expectation operator” $E_j$ by

$$E_j f(x) = 2^j \int_{n2^{-j}}^{(n+1)2^{-j}} f(\xi) d\xi \quad \text{if } x \in [n2^{-j}, (n+1)2^{-j}), \quad n \in \mathbb{Z}.$$  

(1) Show that for $f \in L^1(\mathbb{R})$ we have

$$\lim_{j \to \infty} E_j f(x) = f(x) \text{ almost everywhere.}$$

(2) Show that for $f \in L^2(\mathbb{R})$ one has

$$\lim_{j \to \infty} \|E_j f - f\|_{L^2} = 0.$$  

Problem VI. Let $f \in L^2(\mathbb{R})$, $g \in L^2(\mathbb{R})$. Show that for every $x \in \mathbb{R}$ the convolution integral

$$A(x) = \int f(x - y)g(y) dy$$

is well defined and that $A$ is in fact a continuous function satisfying

$$\lim_{|x| \to \infty} A(x) = 0.$$
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Problem VII.  
(1) Let \( E \subset L^2(\mathbb{R}) \) be a linear subspace which is translation invariant, i.e. if \( f \in E \), then for any \( x \in \mathbb{R} \) the function \( f_x(t) \overset{\text{def}}{=} f(x+t) \) also belongs to \( E \).  
Prove: if \( \dim E < \infty \) then \( E \) is trivial, i.e. \( E = \{0\} \).
(2) Let \( L^2(\mathbb{R}/\mathbb{Z}) \) be the space of all functions in \( L^2_{\text{loc}} \) which are periodic with period 1, and suppose \( E \subset L^2(\mathbb{R}/\mathbb{Z}) \) is a finite dimensional, translation invariant subspace. Must \( E = \{0\} \) hold?

Problem VIII. Denote the unit disc in \( \mathbb{R}^2 \) by \( \Omega \). Let
\[
f(x,y) = \begin{cases} 
1 - x^2 - y^2 & \text{for } (x,y) \in \Omega \\
0 & \text{on } \mathbb{R}^2 \setminus \Omega.
\end{cases}
\]
Prove that in the sense of distributions one has
\[
\Delta f + 4 \chi_\Omega \geq 0,
\]
i.e. show that
\[
\langle \Delta f + 4 \chi_\Omega, \psi \rangle \geq 0
\]
for any test function \( \psi \in C^\infty_c(\mathbb{R}^2) \) with \( \psi(x,y) \geq 0 \) everywhere.  
(Here \( \chi_\Omega \) is the characteristic function of the set \( \Omega \).)

Problem IX. Let \( N \geq 3 \). Show that there is one and only one value of \( p \in [1, \infty) \) such that for all \( f \in C^\infty_c(\mathbb{R}^N) \) one has
\[
\left( \int_{\mathbb{R}^N} |f(x)|^p \, dx \right)^{1/p} \leq C \sum_{i,j=1}^N \int_{\mathbb{R}^N} \left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| \, dx
\]
where \( C \) is a constant which does not depend on \( f \).
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Problem VII.

(1) Let $F$ and $G$ be functions defined on $\mathbb{C}$, assume that

$$F(z) = z^{10} + z^9 + f(z), \quad G(z) = z^{10} + 9z^9 + g(z),$$

where $f$ and $g$ are continuous, but not necessarily holomorphic, functions such that for some constant $C > 0$,

$$|f(z)| + |g(z)| \leq C(1 + |z|^8), \quad \text{for all } z \in \mathbb{C}.$$

Find the limit as $R \to \infty$ of

$$\int_{|z|=R} \frac{F(z)}{G(z)} \, dz$$

(integral over the circle $|z| = R$, with counterclockwise orientation).

(2) For all values of $R$ for which it makes sense, evaluate:

$$\int_{|z|=R} \frac{z^{10} + z^9 + 1}{z(z + 1)^9} \, dz.$$

Problem VIII. Let $\Delta$ be the open unit disc in $\mathbb{C}$, and $\Delta_{\frac{1}{2}}$ be the open disc of radius $\frac{1}{2}$, with center at 0. Let $h$ be a holomorphic map from $\Delta$ into itself.

(1) Assume that $h(0) \in (-1, 0)$. Show that $\frac{1}{2} \not\in h(\Delta_{\frac{1}{2}})$.

(2) Assume that $h(\Delta_{\frac{1}{2}}) \supset \Delta_{\frac{1}{2}}$. Show that there exists $\theta \in \mathbb{R}$, such that $h(z) \equiv e^{i\theta}z$.

Problem IX.

(1) Let $u$ be a harmonic function defined on a neighborhood $V$ of $[-i, +i]$ in $\mathbb{C}$. Assume that $u(x) = 0$ for all $x \in V \cap \mathbb{R}$. Show that $u(-i) = -u(i)$.

(2) Let $\varphi$ be the map defined on $\mathbb{C}$ by $\varphi(z) = z + ix^2$. Describe $\varphi([-i, +i])$.

Let $w$ be a harmonic function defined on a neighborhood $V_1$ of $[-2i, \frac{1}{4}]$. Show that if $w(x + ix^2) = 0$ if $x \in \mathbb{R}$ and $x + ix^2 \in V_1$ (i.e. $w$ vanishes on the parabola $y = x^2$), then $w(-2i) = 0$.

(3) Give examples of harmonic functions defined either on $\mathbb{C}$, or just on a neighborhood of 0, that vanish identically on the parabola $y = x^2$. 
Problem I  Prove or disprove the following
(a) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for $n = 1, 2, \ldots$, then $\sum_{n=1}^{\infty} a_n^3$ converges.
(b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^3$ converges.

Problem II  Show that
\[ \int_{0}^{\infty} e^{-tx} \sin \frac{x}{t} \frac{dx}{x} = \frac{\pi}{2} - \arctan t, \quad t > 0. \]
*(Justify all steps.)*

Problem III  
(a) Let $f$ be a differentiable function defined on $[-1, 1]$. Assume that $|f'| = |f|^{1/2}$. Prove that if $f(0) > 0$ then $f(1) > 1/4$ and that if $f(0) < 0$ then $f(-1) < -1/4$.
(b) Let $\epsilon > 0$. Find a differentiable function $g$ defined on $[-1, 1]$ such that $g'(x) = x|g(x)|^{1/2}, g(0) \neq 0$ but $|g(x)| \leq \epsilon$ for $x \in [-1, 1]$.

Problem IV  Let $p \in [1, \infty)$. For $f \in L^p(\mathbb{R})$ define the functions
\[ g_n(x) = \frac{1}{n} \sum_{k=1}^{n} f \left( x + \frac{k}{n} \right). \]
Show that the sequence $g_n$ converges in $L^p(\mathbb{R})$, and determine the limit function.

Problem V  Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function which vanishes for $x^2 + y^2 > R^2$.
(a) Show that for every $\theta \in [0, 2\pi]$ one has
\[ |f(0,0)| \leq \int_{0}^{\infty} |\nabla f(r \cos \theta, r \sin \theta)| \, dr. \]
(b) Let $p > 2$. Show that there exists $C_{p,R} > 0$ (depending only on $p$ and $R$) such that
\[ |f(0,0)| \leq C_{p,R} \left( \int_{\mathbb{R}^2} |\nabla f|^p \, dx \, dy \right)^{1/p}. \]
*(Hint: Integrate the inequality from part (1) over all $\theta \in [0, 2\pi]$.)
(c) Show that there is no constant $C < \infty$ such that
\[ |f(0,0)| \leq C \int_{\mathbb{R}^2} |\nabla f| \, dx \, dy \]
for all continuously differentiable $f : \mathbb{R}^2 \to \mathbb{R}$ which vanish for $x^2 + y^2 > 1$.

Problem VI  Assume that $(\Omega, \Sigma, \mu)$ is a measure space with $\mu(\Omega) < \infty$. A sequence $f_n$ of complex measurable functions is said to converge in measure to a complex measurable function $f$, if for every $\epsilon > 0$ there exists $N$ such that
\[ \mu(\{ x : |f_n(x) - f(x)| > \epsilon \}) < \epsilon, \quad \text{for } n \geq N. \]
Prove or disprove (with a counter-example) the following statements:
(a) If $f_n \to f$, a.e. then $f_n \to f$ in measure.
(b) If $f_n \to f$ in $L^p$, with $1 \leq p \leq \infty$, then $f_n \to f$ in measure.
(c) If $f_n$ is a sequence in $L^2$ such that for every $g \in L^2$, $\int_{\Omega} f_n g \to 0$ as $n \to \infty$, then $f_n \to 0$ in measure.
Problem VII

(a) (Formula for integration by parts) Let \( u, v \) be smooth functions on \( \mathbb{R}^2 \). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) whose boundary is piecewise smooth. Define \( R_1, R_2 \) by

\[
\int_{\Omega} u(x, y) \frac{\partial v}{\partial x}(x, y) \, dx \, dy = - \int_{\Omega} v(x, y) \frac{\partial u}{\partial x}(x, y) \, dx \, dy + R_1,
\]
\[
\int_{\Omega} u(x, y) \frac{\partial v}{\partial y}(x, y) \, dx \, dy = - \int_{\Omega} v(x, y) \frac{\partial u}{\partial y}(x, y) \, dx \, dy + R_2.
\]

Express \( R_1 \) and \( R_2 \) by integrals on the boundary \( \partial \Omega \).

(b) Let \( Q = \{(x, y) : y \geq |x|, x \leq 0 \} \). Let \( \varphi \) be a smooth function on \( \mathbb{R}^2 \) with compact support. Show that

\[
\int_{Q} (y + x) \left( \frac{\partial^2 \varphi(x, y)}{\partial y^2} - \frac{\partial^2 \varphi(x, y)}{\partial x^2} \right) \, dx \, dy = \int_{0}^{\infty} \varphi(0, y) \, dy - \int_{0}^{\infty} y \frac{\partial \varphi}{\partial x}(0, y) \, dy.
\]

(c) Consider the function

\[
f(x, y) \overset{\text{def}}{=} \begin{cases} 
 y - |x| & \text{for } y \geq |x| \\
 0 & \text{otherwise}
\end{cases}
\]

Show that \( S = \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2} \) is a nonnegative distribution, i.e., show that \( \langle S, \varphi \rangle \geq 0 \) for any nonnegative test function \( \varphi \in \mathcal{D}(\mathbb{R}^2) \).

Problem VIII

Let \( E \) be a Banach space whose closed unit ball will be denoted by \( \overline{B} \). A linear operator \( T \) from \( E \) into itself is said to be a compact operator if \( T(\overline{B}) \) is a relatively compact subset of \( E \).

(a) Give a non-trivial example of compact operators and give an example of application.

(b) Assume that \( E \) is a Hilbert space. Show that if \( T \) is a compact operator then \( T(\overline{B}) \) is a compact subset of \( E \) (not only relatively compact).

(Hint: Use weak convergence.)

(c) Find a more general hypothesis on \( E \) in order that the conclusion in (b) still holds.

Problem IX

(a) Let \( C_0^\infty(\mathbb{R}) \) be the set of smooth functions on \( \mathbb{R} \) with compact support. Let \( \varphi \in C_0^\infty(\mathbb{R}) \).

Assume that \( g \in C^3(\mathbb{R}) \) and that \( g(0) = g'(0) = g''(0) = g'''(0) = 0 \). Set \( g_k(x) = g(x)\phi(kx) \).

Prove that \( g_k \) and its derivatives of order \( \leq 3 \) tend to 0 uniformly on \( \mathbb{R} \) as \( k \to \infty \).

(b) What are all the distributions \( T \) on \( \mathbb{R} \) supported by \( \{0\} \), and such that for some constant \( K \) and for all \( \varphi \in C_0^\infty(\mathbb{R}) \)

\[
|T\varphi| \leq K \sup_{x \in [-1, 1]} \{|\varphi(x)| + |\varphi'(x)| + |\varphi''(x)| + |\varphi'''(x)|\}.
\]

(c) Find a distribution \( S \) on \( \mathbb{R} \) that agrees with the function \( \frac{1}{x^2} \) on \( \mathbb{R} \setminus \{0\} \).

(d) What are all the distributions on \( S \), satisfying (c) and the additional property that, for some constant \( K \) and any \( \varphi \in C_0^\infty(\mathbb{R}) \),

\[
|S\varphi| \leq K \sup_{x \in \mathbb{R}} \{|\varphi(x)| + |\varphi'(x)| + |\varphi''(x)| + |\varphi'''(x)|\}.
\]
Problem VII  Let $0 < \alpha < 1$. Let $f(z)$ be the determination of $z^\alpha$ on $\mathbb{C} \setminus (-\infty, 0]$ with $f(1) > 0$. Let $g(z)$ be the determination of $(1 + z)^{1-\alpha}$ on $\mathbb{C} \setminus (-\infty, 0]$ with $g(1) > 0$.

(a) What are the limits as $t \to 0^+$, and as $t \to 0^-$, of $f(-\frac{1}{2} + it)$ and $g(-\frac{1}{2} + it)$?

(b) Show that $fg$ extends holomorphically to $\mathbb{C} \setminus [-1, 0]$.

(c) Evaluate

$$\int_{-1}^{0} \frac{dx}{z^\alpha(1 + x)^{1-\alpha}}.$$

Problem VIII  Let $f(z)$ be holomorphic on the unit disk in $\mathbb{C}$. Fix $r \in (0, 1)$. Assume that $f(r) = \max(|f(z)|: |z| = r)$.

(a) Show that $f'(r) > 0$, if $f$ is non-constant.

(b) Show that if $f(0) = 0$, then $f'(r) \geq \frac{f(r)}{r}$ and equality holds if and only if $f(z) = cz$ for some non-negative constant $c$.

Problem IX

(a) Show that there is no holomorphic function $f(z)$ on $\{z \in \mathbb{C}; 1 < |z| < 3\}$ satisfying

$$\left| \frac{f(z)^2}{z} - 1 \right| < 1.$$

(b) Show that there exists $\epsilon > 0$ so that no holomorphic function on $\{z \in \mathbb{C}; 1 < |z| < 3\}$ satisfies

$$\left| \frac{|f(z)|^2}{|z|} - 1 \right| < \epsilon.$$
ADVANCED CALCULUS

Problem I
Let $U \subset \mathbb{R}^2$ be a nonempty open subset.
Use differential calculus to show that a continuously differentiable map $f : U \rightarrow \mathbb{R}^2$ cannot be injective.

Problem II
Let $\varepsilon_n > 0$ be a sequence with $\sum_{n=1}^{\infty} \varepsilon_n < \infty$.
(a) Suppose that $u_n$ is a sequence of real numbers satisfying
$$u_{n+1} \leq u_n + \varepsilon_n$$
for all $n \geq 1$. Show that $\lim_{n \rightarrow \infty} u_n$ exists (the possibility $\lim_{n \rightarrow \infty} u_n = -\infty$ is allowed.)

(b) Suppose that $u_n > 0$ are real numbers satisfying $u_n \leq 1 + \varepsilon_n$. Show that $\lim_{n \rightarrow \infty} \prod_{k=1}^{n} v_k$ exists.

Problem III
According to a Theorem of Weierstrass, every continuous function on $[-1, +1]$ can be uniformly approximated by a sequence of polynomials. Here we study the question of approximation by polynomials of fixed degree.

Let $f$ be a $C^4$ function defined on $[-1, +1]$ (i.e. $f$ and its derivatives of order $\leq 4$ are continuous functions on $[0, 1]$.) Show that there is a constant $C > 0$ such that for every polynomial $P$ of degree $\leq 4$
$$\sup_{|x| \leq 1} |f(x) - P(x)| \geq C \left| \int_{-1}^{+1} x(x^2 - 1)^4 f^{(4)}(x) \, dx \right|.$$
Either give an explicit value of $C$ or indicate very clearly an easy computation that would lead to such a value. Give full justifications.

7.2.1

Problem IV
Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function which vanishes outside the unit circle and define
$$I(f) = \int_{0}^{2\pi} \int_{0}^{1} f(r \cos \theta, r \sin \theta) \, dr \, d\theta.$$ For which $p \geq 1$ is there a constant $C_p$ such that
$$I(f) \leq C_p \|f\|_{L^p(\mathbb{R}^2)}$$
for all such functions $f$.

Problem V
Let $f \in L^\infty(\mathbb{R})$ with $f(x+1) = f(x)$.
(a) Show that for every measurable subset $E \subset [0, 1]$ we have
$$\lim_{n \rightarrow \infty} \int_{E} f(n x) \, dx = |E| \int_{0}^{1} f(x) \, dx.$$ Hint: one approach is to first show that it is true for the functions $f_k(x) = e^{2\pi ikx}$, $k = 0, \pm 1, \pm 2 \ldots$.

(b) Suppose that there is a measurable set $E \subset [0, 1]$ with $|E| > 0$ such that for some sequence of integers $n_k \rightarrow \infty$,
$$\lim_{k \rightarrow \infty} f(n_k x) = g(x)$$
exists for all $x \in E$. Show that there is a constant $C$ such that $f(x) = C$ almost everywhere on $[0, 1]$.

Hint: First use part (a) to show that there is a constant $C$ such that $g(x) = C$ almost everywhere on $E$. Then use part (a) again (with a different $f$) to show that $f(x) = C$ almost everywhere on $[0, 1]$.
Problem VI

(a) True or false? In other words, prove or disprove the following statements:
(i) If $\mu$ is a finite Borel measure on $\mathbb{R}$, then $\lim_{x \searrow x_0} \mu((-\infty, x]) = \mu((-\infty, x_0])$ holds for any $x_0 \in \mathbb{R}$.
(ii) If $\mu$ is a finite Borel measure on $\mathbb{R}$, then $\lim_{x \nearrow x_0} \mu((-\infty, x]) = \mu((-\infty, x_0])$ holds for any $x_0 \in \mathbb{R}$.

Let $E \subset \mathbb{R}$. Let $f$ be a continuous function on $\mathbb{R}$. Assume that $f$ is differentiable at any point $x \in \mathbb{R} \setminus E$, and that for any such point $f'(x) = 0$.

(b) Assume that the set $E$ has Lebesgue measure 0, must $f$ be constant?
(c) Assume that the set $E$ is countable. Show that $f$ is constant. Although the result is true in this generality, full credit will be given for a proof in the special and easier case when $E$ is a closed countable set.

Problem VII

Show that there is a distribution $U$ on $\mathbb{R}$ such that for all test functions $\varphi \in C_0^\infty(\mathbb{R})$ that vanish identically near 0:

$$\langle U, \varphi \rangle = \int_{\mathbb{R}} \frac{\varphi(x)}{x^4} \, dx.$$ 

Show that there is no distribution $V$ on $\mathbb{R}$ such that for all test functions $\varphi \in C_0^\infty(\mathbb{R})$ that vanish identically near 0:

$$\langle V, \varphi \rangle = \int_{\mathbb{R}} \varphi(x)e^x \, dx.$$ 

Problem VIII

For $f \in L^2(\mathbb{R})$ let

$$Tf(x) = \int_0^1 f(x+y) \, dy.$$ 

(a) Show that $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.
(b) Show that $\|Tf\|_2 \leq \|f\|_2$, and equality holds if and only if $f = 0$ almost everywhere.
(c) Prove or disprove that the map $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by $S : f \mapsto f - Tf$ is onto.

Problem IX

Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions whose derivatives $f'_n$ in the sense of distributions belong to $L^2(0, 1)$. The functions also satisfy $f_n(0) = 0$.

(a) Assume that

$$\lim_{n \to \infty} \int_0^1 f'_n(x)g(x) \, dx$$

exists for all $g \in L^2(0, 1)$. Show that the $f_n$ converge uniformly as $n \to \infty$.

(b) Assume that

$$\lim_{n \to \infty} \int_0^1 f'_n(x)g(x) \, dx$$

exists for all $g \in C([0, 1])$. Is it still necessarily true that the $f_n$ converge uniformly?
Problem VI

(a) True or false? In other words, prove or disprove the following statements:

(i) If $\mu$ is a finite Borel measure on $\mathbb{R}$, then $\lim_{x \to x_0} \mu((\infty, x]) = \mu((\infty, x_0])$ holds for any $x_0 \in \mathbb{R}$.

(ii) If $\mu$ is a finite Borel measure on $\mathbb{R}$, then $\lim_{x \to x_0} \mu((\infty, x]) = \mu((\infty, x_0])$ holds for any $x_0 \in \mathbb{R}$.

Let $E \subseteq \mathbb{R}$. Let $f$ be a continuous function on $\mathbb{R}$. Assume that $f$ is differentiable at any point $x \in \mathbb{R} \setminus E$, and that for any such point $f'(x) = 0$.

(b) Assume that the set $E$ has Lebesgue measure 0, must $f$ be constant?

(c) Assume that the set $E$ is countable. Show that $f$ is constant. Although the result is true in this generality, full credit will be given for a proof in the special and easier case when $E$ is a closed countable set.

7 2 2

Problem VII Let $f_n(z)$ be a sequence of polynomials. Assume that for some function $h : \mathbb{C} \to \mathbb{C}$ one knows that

$$\lim_{n \to \infty} f_n^2(z) + f_n(z) = h(z)$$

uniformly on each compact subset of $\mathbb{C}$.

(a) Show that $h(z)$ is not the polynomial $h(z) = z$.

(b) If $h(z) = az^2 + bz + c$, find all possible values of $a, b, c$ (or a necessary and sufficient condition on $a, b, c$).

Problem VIII Suppose that $f$ is holomorphic in the unit disc in $\mathbb{C}$ and that

$$\int_0^{2\pi} |f(re^{it})|^p dt \leq \frac{C}{(1 - r)^A}$$

for some $1 < p < \infty$ and some constants $C > 0$ and $A > 0$. Show that $|f(z)| \leq \frac{D}{(1 - |z|)^B}$ for some positive constants $D$ and $B$. Try to find the best value of $B$.

Problem IX Let $g$ be a continuous function defined on the interval $[-1, +1]$ in $\mathbb{R}$. It is a classical result that if one sets

$$g_\tau(x) = \int_{-1}^{+1} \frac{1}{\sqrt{2\pi}} g(t) e^{-\frac{(x-t)^2}{2\tau}} dt,$$

then for any $x \in (-1, +1)$, $g_\tau(x)$ tends to $g(x)$ as $\tau \to 0^+$, and the convergence is uniform on smaller intervals.

(It is just a matter of classical approximate identity kernels, it shows up naturally when solving the heat equation and it was used by Weierstrass in proving his approximation theorem.)

Now, let $g$ be a holomorphic function defined on $\mathbb{C}$. For $z \in \mathbb{C}$ and $\tau > 0$ set:

$$g_\tau(z) = \int_{-1}^{+1} \frac{1}{\sqrt{2\pi}} g(t) e^{-\frac{(z-t)^2}{2\tau}} dt.$$

(a) Prove that $g_\tau$ is an entire function, i.e. a holomorphic function defined on all of $\mathbb{C}$.

(b) Find a region $U \subseteq \mathbb{C}$ containing a neighborhood of 0 such that for all $z \in U$, $g_\tau(z)$ tends to $g(z)$ as $\tau \to 0^+$.

Hint: If $z = x + iy$, switch from integration on $[-1, +1]$ to integration on the line segment $[-1 + iy, 1 + iy]$.
Problem I
(a) Show that if \( \sum a_n \) converges then there exists a sequence \( b_n \to \infty \) so that \( \sum a_nb_n \) is still convergent.

(b) Let \( b_n \) be an unbounded sequence. Show that there exists a convergent \( \sum a_n \) so that \( \sum a_nb_n \) is divergent.

Problem II
(a) For which real values of \( p \) does the integral
\[
I_p = \int_0^\infty \int_0^\infty \frac{dx
dy}{1 + x^2 + y^p + x^2y^2}
\]
converge?
(b) For which real values of \( p \) does the sum
\[
\sum_{m=0}^\infty \sum_{n=0}^\infty \frac{1}{1 + m^2 + n^p + m^2n^2}
\]
converge?

Problem III Let \( a_n \) be a convergent sequence. Define
\[
F(\lambda) = \sum_{n=1}^\infty \lambda e^{-\lambda n}a_n, \quad \lambda > 0.
\]
(a) Show that \( \lim_{\lambda \to 0} F(\lambda) \) exists.

(b) Show that \( F : (0, \infty) \to \mathbb{R} \) is a continuously differentiable function.

(c) Does the limit
\[
\lim_{\lambda \to 0} \frac{F(\lambda) - F(0)}{\lambda}
\]
exist? (Here we interpret \( F(0) \) as \( \lim_{\lambda \to 0} F(\lambda) \).)

Problem IV For \( f \in L^1_{\text{loc}}(\mathbb{R}^3) \), define
\[
(Kf)(x) = \int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} |f(y)|
dy, \quad x \in \mathbb{R}^3
\]
and
\[
\|f\|_K = \|Kf\|_{L^1(\mathbb{R}^3)}.
\]
Let \( X = \{f \in L^1_{\text{loc}}(\mathbb{R}^3) : \|f\|_K < \infty \} \).

(a) Prove or disprove that the normed vector space \( (X, \| \cdot \|_K) \) is complete.

(b) For which \( p \in [1, \infty) \) is \( L^p(\mathbb{R}^3) \) a subset of \( X \)?

Problem V Let \( K \) be a non-empty compact subset of \( \mathbb{R} \), of Lebesgue measure 0.

(a) Show that there is a continuous non-negative function \( f \) such that \( f = 0 \) on \( K \) and \( f > 0 \) off \( K \), and such that \( \int_{\mathbb{R}} \frac{dx}{f(x)} < +\infty \).

(b) Show that there is no function \( f \) as above that is continuously differentiable.
Problem VI  Let $f \in L^2(\mathbb{R})$.

Are the following true statements? Prove or disprove by a counter example.

(a) If $f$ is continuous then $f(x)$ tends to 0 as $|x|$ tends to $+\infty$.

(b) $\int_{n}^{n+1} |f(t)| \, dt$ tends to 0 as $n \to \infty$.

(c) $\sqrt{n} \int_{n}^{n+1} |f(t)| \, dt$ tends to 0 as $n \to \infty$.

(d) $\liminf_{n \to \infty} \sqrt{n} \int_{n}^{n+1} |f(t)| \, dt = 0$.

Problem VII  Let $S(\mathbb{R}^d)$ be the set of smooth functions $f$ satisfying

$$
\rho_N(f) = \sup_{|\alpha| \leq N, x \in \mathbb{R}^d} (1 + |x|^2)^N |D^\alpha f(x)| < \infty, \quad N = 0, 1, 2, \ldots,
$$

where $D^{\alpha} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$. $S(\mathbb{R}^d)$ is a vector space, in which the topology is defined by semi-norms $\rho_N, N = 0, 1, 2, \ldots$. Let $S'(\mathbb{R}^d)$ be the set of continuous linear functionals on $S(\mathbb{R}^d)$. Recall that each element in $S'(\mathbb{R}^d)$ is a distribution on $\mathbb{R}^d$.

Let $T \in S'(\mathbb{R}^2)$ satisfy

$$
D^{\alpha} T = 0,
$$

for any multi-index $\alpha$ with $|\alpha| = 2$. Show that $T$ is a polynomial in $x$ of degree at most 1. (The statement is true, under the assumption that $T$ is a distribution. But you don’t have to prove it.)

Problem VIII  Construct a function $f$ in $L^2(\mathbb{R}^2)$, with distribution derivative $f'$ in $L^2(\mathbb{R}^2)$, that diverges to infinity ($\lim_{x \to r} |f(x)| = \infty$) at every rational point $r$ in the unit square.

Problem IV  A subset $S$ of a (complex) Banach space $X$ is called weakly bounded if $\sup_{x \in S} |\lambda(x)| < \infty$ for any $\lambda \in X^*$. The set $S$ is called strongly bounded if $\sup_{x \in S} ||x|| < \infty$. Prove that $S$ is strongly bounded if and only if $S$ is weakly bounded.
Problem VI  Let \( f \in L^2(\mathbb{R}) \).
Are the following true statements? Prove or disprove by a counter example.
(a) If \( f \) is continuous then \( f(x) \) tends to 0 as \( |x| \) tends to \(+\infty\).
(b) \( \int_{n}^{n+1} |f(t)| \, dt \) tends to 0 as \( n \to \infty \).
(c) \( \sqrt{n} \int_{n}^{n+1} |f(t)| \, dt \) tends to 0 as \( n \to \infty \).
(d) \( \liminf_{n \to \infty} \sqrt{n} \int_{n}^{n+1} |f(t)| \, dt = 0 \).

Problem VII
(a) Show that there is no holomorphic function \( g(z) \) on \( \mathbb{C} \setminus [-2, 2] \) satisfying
\[ g(z)^3 - 3g(z) = z, \quad z \in \mathbb{C} \setminus [-2, 2]. \]
(b) Let \( D = \mathbb{C} \setminus ((-\infty, -2] \cup [2, \infty)) \). It is a fact that there is a holomorphic function \( f(z) \) on \( D \) satisfying
\[ f(z)^3 - 3f(z) = z, \quad z \in D, \quad f(0) = 0. \]
(You can accept this fact.) Prove that \(-f(-z) = f(z)\) holds on \( D \).
(c) For \( \epsilon > 0 \) let \( \gamma_\epsilon \) be the oriented path consisting of: the horizontal half line connecting \(+\infty - i\epsilon \) to \(2 - \epsilon - i\epsilon \), the vertical segment connecting \(2 - \epsilon - i\epsilon \) to \(2 - \epsilon + i\epsilon \), and the horizontal half line connecting \(2 - \epsilon + i\epsilon \) to \(+\infty + i\epsilon \).
For the function \( f \) in (b), evaluate
\[ \lim_{\epsilon \to 0^+} \int_{\gamma_\epsilon} \frac{dz}{z^2 f(z)}. \]

Problem VIII  Let \( u(z) \) be a real harmonic function on \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \).
(a) Show that there is a constant \( c \) so that \( u(z) - c \log |z| = f(z) + \overline{f(z)} \) for some holomorphic function \( f \) defined on \( \mathbb{C}^* \).
(b) Show that \( u(z) \) has a finite limit as \( z \to 0 \), if
\[ \lim_{z \to 0} \frac{u(z)}{\log |z|} = 0. \]
(c) Show that \( u(z) \) has a finite limit as \( |z| \to \infty \), if
\[ \lim_{|z| \to \infty} \frac{u(z)}{\log |z|} = 0. \]

Problem IX  Let \( D \) be a domain in \( \mathbb{C} \), with non-empty boundary \( \partial D = (\overline{D} \cap \mathbb{C}) \setminus D \). Let \( f(z) \) be holomorphic on \( D \) and continuous on \( \overline{D} \cap \mathbb{C} \). Assume that there are two constants \( A \) and \( B \) so that
\[ \sup_{z \in \partial D} |f(z)| \leq A, \quad \sup_{z \in D} |f(z)| \leq B. \]
Show that \( |f(z)| \leq A \) on \( D \).
Hint: Take \( a \in \partial D \) and consider
\[ \frac{f(z)^n}{z-a}, \quad z \in D \setminus \{z \in |z-a| \leq \epsilon\}, \quad n = 1, 2, 3, \ldots. \]
Problem I  Assume that $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are sequences of nonnegative real numbers such that

(i) $a_n \leq a_{n+1}$ for any $n = 1, 2, \ldots$.
(ii) $b_n \geq b_{n+1}$ for any $n = 1, 2, \ldots$, and $\lim_{n \to \infty} b_n = 0$.
(iii) $\sum_{n=1}^\infty a_n(b_n - b_{n+1})$ is convergent.

(a) Prove that $\lim_{n \to \infty} a_nb_n = 0$.
(b) Show that conclusion (a) may fail if assumption (i) is omitted.

Problem II  Let

$$F(t) = \int_0^\infty e^{-x^2} \cos(tx) \, dx, \quad t \in \mathbb{R}.$$ 

(a) Prove that $F'(t) = -\frac{1}{2}F(t)$. (Justify all steps.)
(b) Find $F(1)$.

Problem III  Prove that

$$xy + x^4 - y^4 = 0$$

admits a continuous solution $y = f(x)$ for $|x| < \frac{1}{100}$.

Problem IV  Let

$$F(x) = \int_{[0,\infty)} \frac{e^{-xt}}{|\sin t|^a} \, dt, \quad x > 0.$$ 

(a) Find the values of the parameter $a$ for which the function $F$ is well-defined on $(0, \infty)$ (i.e. the integrand is in $L^1$).
(b) Show that $F \in C^\infty(0, \infty)$ for these values of $a$. (Hint: Use the Lebesgue Dominated Convergence Theorem.)

Problem V  Assume that $(\Omega, \Sigma, \mu)$ is a measure space and $f \in L^p(\Omega)$ for some $0 < p < \infty$.

(a) Show that

$$\lim_{q \to \infty} \|f\|_{L^q} = \|f\|_{L^\infty}.$$ 

(b) Does the conclusion (*) still hold if we omit the assumption $f \in L^p(\Omega)$ (proof or counterexample)?

Problem VI  Construct a sequence of continuous functions $f_n$ on $[0, 1]$ such that $0 \leq f_n \leq 1$,

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0,$$

but the sequence $f_n(x)$ does not converge for any $x \in [0, 1]$. 
Problem VII  Recall that if $T$ is a distribution on $\mathbb{R}^2$, $(\frac{\partial}{\partial x} T)(\varphi) = -T(\frac{\partial}{\partial x} \varphi)$, for any smooth function $\varphi$ with compact support.

Let $D \subset \mathbb{R}^2$ be the domain $x > |y|$. Let $\chi_D$ be the characteristic function of $D$. (So $\chi_D = 1$ on $D$ and $\chi_D = 0$ on $\mathbb{R}^2 \setminus D$.) Let $\varphi$ be a smooth function on $\mathbb{R}^2$ with compact support. Assume $\varphi \geq 0$ and $\varphi(0) > 0$. Let $\varphi_n(x, y) = \varphi(nx, ny)$. Determine values of $\alpha$ for which $\lim_{n \to \infty} (\frac{\partial \chi_D}{\partial x})(n^\alpha \varphi_n)$ exists (and is finite).

Problem VIII  Let $f$ be a continuous function on $\mathbb{R}^2$. Assume that the distributional partial derivative $\frac{\partial f}{\partial x}$ is in $L^\infty_{\text{loc}}(\mathbb{R}^2)$, that is that for some $a \in L^\infty_{\text{loc}}(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} f \frac{\partial \varphi}{\partial x} \, dx \, dy = - \int_{\mathbb{R}^2} a \varphi \, dx \, dy$$

for all smooth functions $\varphi$ on $\mathbb{R}^2$ with compact support. Prove that the distributional derivative $\frac{\partial}{\partial x}(f(x, 0))$ is in $L^\infty_{\text{loc}}(\mathbb{R})$.

Problem IX  Let $0 < \alpha \leq 1$. Let $\text{Lip}_\alpha$ be the set of functions $f$ on $[0, 1]$ satisfying

$$|f|_\alpha = \sup_{0 \leq x \leq 1} |f(x)| + \sup_{0 \leq x < y \leq 1} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$ 

Let $S \subset \text{Lip}_\alpha$ be a closed linear subspace of $L^2([0, 1])$.

(a) Prove that there is a positive constant $c$ such that

$$|f|_\alpha \leq c\|f\|_{L^2}, \quad f \in S.$$ 

(b) Prove that $S$ is finite-dimensional.
Problem VII  Let $\Delta$ be the unit disc in $\mathbb{C}$.
(a) Find a sequence of holomorphic functions $f_n: \Delta \to \mathbb{C} \setminus \{0\}$ such that $f_n(0) = \frac{1}{2}$ and $\lim_{n \to \infty} f_n(\frac{1}{2}) = 0$.
(b) Prove that if $f: \Delta \to \mathbb{C} \setminus \{0\}$ is holomorphic and $f(0) = \frac{1}{2}$, then $|f(\frac{1}{2})| > c > 0$ for some constant $c$ independent of $f$.

Problem VIII  Evaluate
\[ \int_0^\infty \frac{\log x}{x^3 + 1} \, dx. \]
(Justify all steps.)

Problem IX  Let $f(z)$ be holomorphic on $\text{Re} \, z > 0$ and continuous on $\text{Re} \, z \geq 0$. Assume that $|f(z)| < e^{-|z|}$. Show that $f(z) \equiv 0$. (Hint: Consider $F(z) = f(z)e^{\frac{1}{n(1-\alpha)}} z^\alpha$ as $\alpha$ tends to $1^-$.)
Problem I
(a) Determine all values of $\alpha, \beta$ such that the (possibly) improper integral
$$\int_0^1 x^\alpha \sin(x^\beta) \, dx$$
converges.
(b) Determine all values of $\alpha, \beta$ such that the improper integral
$$\int_0^1 x^\alpha |\sin(x^\beta)| \, dx$$
converges.

Problem II
(a) Suppose $f \in C^2((\varepsilon, \varepsilon))$. Define
$$F(x, y) = \begin{cases} \frac{f(x) - f(y)}{x - y}, & x \neq y, \\ f'(x), & x = y. \end{cases}$$
Show that $F \in C^1((\varepsilon, \varepsilon) \times (\varepsilon, \varepsilon))$.
(b) Suppose $f \in C^2((\varepsilon, \varepsilon))$ and $f(0) = f'(0) = 0$ and $f''(0) = 1$. Show that there exist $\delta, \eta > 0$ and $\varphi \in C^1((\delta, \varepsilon))$ such that $|\varphi(x)| < \eta$ for $|x| < \delta$ and $\varphi'(0) = -1$ and $f(\varphi(x)) = f(x)$.
(c) Show that the above $\varphi$ satisfies $\varphi(\varphi(x)) = x$ for $|x| < \eta'$ and for some $\eta' \in (0, \eta]$.

Problem III
Let $a_k$ be a sequence of non-negative numbers satisfying $0 < \sum_{k=1}^\infty a_k < \infty$. Show that
$$\lim_{x \to +\infty} \sum_{k=1}^\infty a_k \sin \frac{x}{k}$$
does not exist.

Problem IV
Show that if $f \in L^1(\mathbb{R})$ then
$$\lim_{\lambda \to +\infty} \int_{\mathbb{R}} f(x)e^{-i\lambda x} \, dx = 0.$$

Problem V
Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $1 \leq p \leq q \leq r \leq \infty$, and let $L^p = L^p(\Omega)$, $L^q = L^q(\Omega)$, and $L^r = L^r(\Omega)$.
(a) Show that $L^p \cap L^r \subset L^q \subset L^p + L^r$.
   By definition, $L^p + L^r = \{g + h : g \in L^p, h \in L^r\}$.
(b) Define canonical norms on $L^p \cap L^r$, $L^p + L^r$ by
   $$\|f\|_{L^p \cap L^r} = \max\{\|f\|_{L^p}, \|f\|_{L^r}\},$$
   $$\|f\|_{L^p + L^r} = \inf\{\|g\|_{L^p} + \|h\|_{L^r} : f = g + h, g \in L^p, h \in L^r\}.$$
   (You don’t need to verify the above two are norms.)
   Prove that two inclusions in (a) are continuous maps.
Problem VI  Assume \( \{a_n\}, \{b_n\} \in l^2(\mathbb{Z}_+) \), i.e. \( a_1^2 + a_2^2 + \cdots < \infty \) and \( b_1^2 + b_2^2 + \cdots < \infty \).

(a) Using the Cauchy-Schwarz inequality show that

\[
\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{a_n b_m}{m}
\]

is convergent.

(b) Show that

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n + m}
\]

is convergent.

(c) If instead of assuming \( \{a_n\}, \{b_n\} \in l^2(\mathbb{Z}_+) \) we assume \( a_n \in l^p(\mathbb{Z}_+) \) and \( b_n \in l^{p'}(\mathbb{Z}_+) \), with \( p, p' \in (1, \infty), 1/p + 1/p' = 1 \), prove that (b) still holds.

Problem VII  Suppose that \( f \) is holomorphic in \( \{z = x + iy : y > 0\} \) and that \( \lim_{z \to 0} f(z) = L \) exists (and is finite). Let \( S = \{z = x + iy : y > |x|\} \). Show that

\[
\lim_{S \ni z \to 0} zf'(z) = 0.
\]

Problem VIII

(a) Show that

\[
\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}.
\]

(b) Evaluate

\[
\sum_{n=1}^{\infty} \frac{1}{n^4}.
\]

Problem IX  Show that there is no entire function \( f(z) \) satisfying

\[
|f(z) - e^{x}| \leq 3|z|, \quad z \in \mathbb{C}.
\]
Problem VI  Assume \( \{a_n\}, \{b_n\} \in l^2(\mathbb{Z}_+), \) i.e. \( a_1^2 + a_2^2 + \cdots < \infty \) and \( b_1^2 + b_2^2 + \cdots < \infty. \)

(a) Using the Cauchy-Schwarz inequality show that
\[
\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{a_n b_m}{m}
\]
is convergent.

(b) Show that
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n + m}
\]
is convergent.

(c) If instead of assuming \( \{a_n\}, \{b_n\} \in l^2(\mathbb{Z}_+) \) we assume \( a_n \in l^p(\mathbb{Z}_+) \) and \( b_n \in l^{p'}(\mathbb{Z}_+) \), with \( p, p' \in (1, \infty), \frac{1}{p} + \frac{1}{p'} = 1 \), prove that (b) still holds.

Problem VII  Find all functions \( f \in L^1(\mathbb{R}^d) \) with the property that \( f * f = f \). Find all functions \( f \in L^2(\mathbb{R}^d) \) with the property that \( f * f = f \).

Problem VIII  One says that a distribution \( T \) on \( \mathbb{R} \) has order \( s \) at \( p \in \mathbb{R} \), if \( s \) is the smallest non-negative integer such that there is a neighborhood \( U \) of \( p \) such that
\[
| <T, \varphi> | \leq C \max_{x \in U, \ k \leq s} \left| \frac{d^k \varphi(x)}{dx^k} \right|
\]
for some constant \( C > 0 \) and for all smooth functions \( \varphi \) whose supports are contained in \( U \).

Let \( 0 < b_{n+1} < a_n < b_n \) for \( n = 1, 2, 3, \ldots \) Let \( \chi_{[a_n, b_n]} \) be the characteristic function of \( [a_n, b_n] \) and
\[
f = \sum_{n=1}^{\infty} c_n \chi_{[a_n, b_n]}, \quad c_n \in \mathbb{R}.
\]
Assume that \( f \in L^1(-\infty, \infty). \)

(a) Prove that the distribution derivative \( f' \) has order 0 at 0 if \( \sum_{n=1}^{\infty} |c_n| < \infty. \)

(b) Prove that \( f' \) has order 1 at 0 if \( \sum_{n=1}^{\infty} |c_n| = \infty. \)

Problem IX  Let \( X \) be a Banach space. Assume that \( \{v_1, v_2, \ldots\} \) is a dense subset of the unit ball of \( X \). Define the map \( A : L^1(\mathbb{N}) \to X \) by
\[
A((\alpha_1, \alpha_2, \ldots)) = \sum_{n=1}^{\infty} \alpha_n v_n.
\]

(a) Show that the map \( A \) is well defined, linear, and continuous.

(b) Show that \( A(L^1(\mathbb{N})) = X. \)
**Problem I**  Give an example of a Riemann integrable function $f : [0, 1] \rightarrow [0, 1]$ which has a dense set of discontinuities. Verify all conclusions.

**Problem II**  Let
\[ f(x, y) = \sum_{n=1}^{\infty} \frac{x}{x^2 + yn^2}, \quad y > 0. \]

(a) Show that for each $y > 0$, $g(y) = \lim_{x \to +\infty} f(x, y)$ exists. Evaluate the limit function $g(y)$.

(b) Determine if $f(x, y)$ converges to $g(y)$ uniformly for $y \in (0, \infty)$ as $x \to +\infty$.
(Justify all steps.)

**Problem III**  Let
\[ s_n(x) = \sum_{k=1}^{n} \sin(kx). \]

Show that there exists a constant $C$, independent of $N, x$, such that
\[ \sum_{n=1}^{N} \frac{|s_n(x)|}{n^2} < C, \quad 0 < x < \pi, \quad N = 1, 2, 3, \ldots. \]

(*Hint: Estimate $s_n(x)$ for $n \leq \frac{1}{2}$ and for $n > \frac{1}{2}$ separately.*)

**Problem IV**  Give an example of a sequence $f_k$ such that $f_k$ converges weakly to zero in $L^2[0, 1]$ and strongly to zero in $L^{3/2}[0, 1]$, but does not converge strongly in $L^2[0, 1]$. Verify all conclusions.

**Problem V**  Fix a function $g \in L^1(\mathbb{R})$ such that $\int g(x) \, dx = 0$. Denote $g_\epsilon(x) = \epsilon^{-1} g(\epsilon^{-1} x)$. Consider an operator
\[ T_\epsilon f(x) = \int_{\mathbb{R}} g_\epsilon(y) f(x - y) \, dy. \]

(a) Prove that there exists a constant $C$ such that $\|T_\epsilon f\|_p \leq C\|f\|_p$, for all $\epsilon \neq 0$ and $1 \leq p < \infty$.

(b) Prove that $\lim_{\epsilon \to 0} \|T_\epsilon f\|_p = 0$ for any $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$.

**Problem VI**  Fix $\alpha > 0$.

(a) Suppose that $f_n \in L^\infty(\mathbb{R})$ satisfy $\|f_n\|_{L^\infty(\mathbb{R})} \geq n^{1+\alpha}$, $n = 1, 2, \ldots$. Show that there is a function $g \in L^1(\mathbb{R})$ such that $\lim_{n \to \infty} \|f_n g\|_{L^1(\mathbb{R})} = \infty$.

(b) Prove or disprove that (a) holds when $\alpha = 0$. 
Problem VII  Let \( f(z) = 10z + z^2 + iz^4 \).

(a) Show that for each \( w \) with \( |w| < 8 \), \( f(z) = w \) has a unique solution \( z \) satisfying \( |z| < 1 \).
(b) Show that there exist distinct \( z_1, z_2 \) in \( \{ z : |z| < 2 \} \) such that \( f(z_1) = f(z_2) \).

Problem VIII

(a) Let \( f(z) \) be the branch of \( \sqrt{z(1 - z)} \) on \( \mathbb{C} \setminus [0, 1] \) with \( f(2) = \sqrt{2i} \). Determine the values of
\[
\lim_{y \to 0, y > 0} f\left(\frac{1}{2} + iy\right), \quad \lim_{y \to 0, y < 0} f\left(\frac{1}{2} + iy\right).
\]

(b) Evaluate
\[
\int_0^1 \frac{x^2}{\sqrt{x(1 - x)}} \, dx.
\]
(Justify all steps.)

Problem IX  Let \( \Delta = \{ z : |z| < 1 \} \).

(a) Let \( 0 < a_n < 1 \) such that \( \sum_{n=1}^\infty (1 - a_n) \) is convergent. Show that the limit function
\[
f(z) = \lim_{n \to +\infty} \prod_{k=1}^n \frac{a_k - z}{1 - a_k z}, \quad z \in \Delta
\]
is holomorphic, and that \( f \) has zeros at \( a_n \) only.

(b) Give an example of a bounded holomorphic function \( g \) on \( \Delta \) and a sequence \( b_n \) in \( \Delta \) such that \( g \) has simple zeros at \( b_n, n = 1, 2, \ldots \) and
\[
\lim_{n \to \infty} g'(b_n)(1 - |b_n|) = 0.
\]
Verify all conclusions.
Problem VII

(a) Find all distributions $T \in S'(\mathbb{R}^2)$ with the property that
\[ x_1 T = x_2 T = 0. \]

(b) Give an example of a distribution $T \in S'(\mathbb{R}^2)$ which does not have compact support and has the property that
\[ x_1 x_2 T = 0. \]

Problem VIII  Show that if $L \in S'(\mathbb{R})$ and $\phi \in S(\mathbb{R})$ then for $\psi_x(y) = \phi(x - y)$ the function
\[ f(x) = L(\psi_x) \]
is continuous on $\mathbb{R}$, and
\[ |f(x)| \leq C(1 + |x|)^N \]
for some constants $C$ and $N$.

Problem IX  Assume that $H$ is a Hilbert space, $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal set, and $x \in H$. Find
\[ \inf_{c_1, \ldots, c_n \in \mathbb{C}} ||x - \sum_{i=1}^{n} c_i v_i||, \]
in terms of $\|x\|$ and $(x, v_i), i = 1, \ldots, n$. 
Qualifying Exam in Analysis  
Real and Complex Analysis (Math 721-722) Version  
Wednesday, January 17, 2007

Instructions: Do six of the nine problems. To receive credit on a problem, you must show your work and justify your conclusions. To facilitate grading, please use a separate packet of paper for each question. Use a black pen or #2 pencil (no mechanical pencils please!).

1. Let $X$ be a metric space with metric $d$.
   (i) Define $\rho : X \times X \to \mathbb{R}$ by
   \[ \rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}. \]
   Prove that $\rho$ is a metric on $X$.
   (ii) Show that a subset $U$ of $X$ is open with respect to the metric $d$ if and only if it is open with respect to the metric $\rho$.

2. Let $u : \mathbb{R}^3 \to \mathbb{R}$ denote a smooth function and let $\Delta = \partial^2_x u + \partial^2_y u + \partial^2_z u$ be the Laplacian of $u$.
   Suppose that $\Delta u = 1$ on $\mathbb{R}^3$ and $u(x, y, z) = x^2 y^3$ on the sphere of radius $R$ centered at the origin. Find $u(0, 0, 0)$.

3. Let $I$ be a compact subset of $(0, 2\pi)$. Show that the series
   \[ \sum_{k=1}^{\infty} \frac{\sin(kx)}{k} \]
   converges uniformly on $I$.

4. Let $F$ be a closed set in $\mathbb{R}$ whose complement has finite measure, and let $\delta_F(x)$ denote the distance of $x$ to $F$, i.e. $\delta_F(x) = \inf\{|x - y| : y \in F\}$.
   (i) Prove that $\delta_F$ is Lipschitz continuous, in fact
   \[ |\delta_F(x) - \delta_F(y)| \leq |x - y|. \]
   (ii) Let
   \[ M(x) = \int \frac{\delta_F(y)}{|x - y|^2} \, dy. \]
   Show that $M(x) < \infty$ for almost every $x \in F$.
   Hint: For part (ii) consider the integral $\int_F M(x) \, dx$. 

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5. On the interval $[-1,1]$ consider the standard Banach spaces $L^1$ and $L^2$ with the norms

$$\|f\|_{L^1} = \int_{-1}^{1} |f(x)| \, dx; \quad \|f\|_{L^2} = \left( \int_{-1}^{1} |f(x)|^2 \, dx \right)^{1/2}.$$ 

Let $\{f_j\}_{j=1}^{\infty}$ denote a sequence of functions in $L^2$. Assume that $f_j \geq 0$, $\|f_j\|_{L^1} = 2$, and

$$|\|f_j\|_{L^2} - \sqrt{2}| \leq 2^{-j}.$$ 

Show that $\lim_{j \to \infty} f_j(x) = 1$ for almost every $x \in [-1,1]$.

*Hint:* Write $f_j = 1 + h_j$.

6. Given a sequence of functions $f_n \in L^2(\mathbb{R})$, we say that $f_n$ converges weakly to $f \in L^2$ if

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)g(x) \, dx = \int_{\mathbb{R}} f(x)g(x) \, dx \quad \text{for all } g \in L^2(\mathbb{R}).$$

Find a sequence of bounded, (Borel) measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function $f \neq 0 \in L^2(\mathbb{R})$ with the property that $2f$ is a characteristic function.

7. For $z \in \mathbb{C}$ evaluate

$$\frac{1}{2\pi} \int_{0}^{2\pi} \log |e^{i\theta} - z| \, d\theta.$$ 

*Suggestion:* Treat the easier case $|z| > 1$ first.

8. Let $S = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, -1 < y < 1\}$, and let $f : S \to \mathbb{C}$ be a holomorphic function which satisfies the inequality

$$|f(z)| \leq 1 + |z|^2 \quad \text{for all } z \in S.$$ 

Show that for any $n = 0, 1, \ldots$ there is a constant $C_n$ such that

$$|f^{(n)}(x)| \leq C_n (1 + |x|^2) \quad \text{for all } x \in \mathbb{R}.$$ 

What can you say about the constant $C_n$?

9. Let $E$ denote a compact subset of $\mathbb{R}$ of measure 0 (here measure refers to Lebesgue measure on the real line). Let $f : \mathbb{C} \setminus E \to \mathbb{C}$ be a holomorphic function. Show that if $f$ is bounded on any bounded subset of $\mathbb{C} \setminus E$, then $f$ extends to a holomorphic function on $\mathbb{C}$. 


5. On the interval $[-1,1]$ consider the standard Banach spaces $L^1$ and $L^2$ with the norms
\[ ||f||_{L^1} = \int_{-1}^{1} |f(x)| \, dx; \quad ||f||_{L^2} = \left( \int_{-1}^{1} |f(x)|^2 \, dx \right)^{1/2}. \]
Let $\{f_j\}_{j=1}^{\infty}$ denote a sequence of functions in $L^2$. Assume that $f_j \geq 0$, $||f_j||_{L^1} = 2$, and
\[ ||f_j||_{L^2} - \sqrt{2} \leq 2^{-j}. \]
Show that $\lim_{j \to \infty} f_j(x) = 1$ for almost every $x \in [-1,1]$. 

*Hint:* Write $f_j = 1 + h_j$.

6. Given a sequence of functions $f_n \in L^2(\mathbb{R})$, we say that $f_n$ converges weakly to $f \in L^2$ if
\[ \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)g(x) \, dx = \int_{\mathbb{R}} f(x)g(x) \, dx \quad \text{for all} \quad g \in L^2(\mathbb{R}). \]
Find a sequence of bounded, (Borel) measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function $f \neq 0 \in L^2(\mathbb{R})$ with the property that $2f$ is a characteristic function.

7. Let $g : \mathbb{R}^2 \to \mathbb{R}$,
\[ g(x, y) = \begin{cases} 
    x^2 + y^2 & \text{if } x^2 + y^2 \leq 1; \\
    1 & \text{if } x^2 + y^2 \geq 1.
\end{cases} \]
Find the distribution $(\partial_x^2 + \partial_y^2)g$.

8. For any $m \in \{1, 2, \ldots \}$ let $f_m(x) = |x|^{-m}$ on $\mathbb{R} \setminus \{0\}$.
Find a distribution $T_m$ on $\mathbb{R}$ which agrees with $f_m$ on $\mathbb{R} \setminus \{0\}$, i.e.
\[ T_m(\phi) = \int_{\mathbb{R}} f_m(x)\phi(x) \, dx \quad \text{for any } \phi \in C_0^\infty(\mathbb{R} \setminus \{0\}). \]
Give complete justifications (in particular show that your choice of $T_m$ really defines a distribution).

9. Let $H$ be an infinite dimensional Hilbert space.
   (i) Prove that there is no relatively compact neighborhood of the origin.
   (ii) Let $T : H \to H$ be linear, bounded and surjective, and let $B$ be the closed unit ball centered at the origin. Show that $T(B)$ is not compact.