Problem 1. Let $X = A \cup B$ be a topological space, which is a union of two subspaces $A, B \subseteq X$.

1. Assume that $A$ and $B$ are closed, that the pairs $(X, A), (X, B), (A, A \cap B),$ and $(B, A \cap B)$ have the homotopy extension property, and that $A, B, A \cap B$ are all contractible. Prove that $X$ is contractible.

2. Given an example of $X = A \cup B$ with $A$ open and $B$ closed such that $A, B,$ and $A \cap B$ are contractible, but $X$ is not. Justify your answer.

Proof. For a space $Y$ to be contractible, it means that we have a homotopy between the identity map on $Y$ and the constant map at a point $y_0$; that is, a continuous function $F: Y \times I \to y_0$ such that $F(y, 0) = y$ and $F(y, 1) = y_0$ for all $y \in Y$. Fix $x_0 \in A \cap B$. Since $A \cap B$ is contractible, we have a map $H: A \cap B \times I \to x_0$ so that $H(x, 0) = x$ for $x \in A \cap B$ and $H(x, 1) = x_0$. Also, we know that $A$ is a contractible space by assumption, so there is a map $G_A : A \times I \to x_0$ so that $G(a, 0) = a$ for all $a \in A$ and $G(a, 1) = x_0$. Notice that $G_A|_{A \cap B}(a, 0) = a$ for all $a \in A \cap B$ so $G_A|_{A \cap B}$ is a lift of the homotopy $H$ at time $t = 0$. Since $(X, A)$ has the homotopy extension property, we can lift the homotopy $H$ by a homotopy $\tilde{H}_A : A \times I \to x_0$ so that $\tilde{H}_A|_{A \cap B} = H(x, t)$ for all values of $t$. Similarly, we know that there exists a homotopy $\tilde{H}_B : B \times I \to x_0$ with $\tilde{H}_B(x, 0) = x$ and $\tilde{H}_B(x, 1) = x_0$ such that when we restrict this homotopy to $A \cap B$ we have the homotopy $H$. We now consider the function $G : X \times I \to x_0$ such that $G(x, t) = \tilde{H}_A(x, t)$ if $x \in A$ and $G(x, t) = \tilde{H}_B(x, t)$ if $x \in B$. Note that this function is well defined for $x \in A \cap B$ since both $\tilde{H}_A$ and $\tilde{H}_B$ were created to extend $H$. Now $G$ is a continuous function such that $G(x, 0) = x$ for all $x \in X$ and $G(x, 1) = x_0$, so we see that $X$ is a contractible space.

For part 2, take $X$ to be $S^2$. Fix $x_0$ on the equator of $S^2$ and fix a closed half disk of radius $\delta$ around $x_0$; call this half-neighborhood $U$. Let $B$ be the lower hemisphere of $S^2$ union $U$; let $A$ be the upper hemisphere (plus a radius $\epsilon$ ball further over the equator) minus $U$. Now $B$ is homotopic to $D^2$ and $A$ is homotopic to an open disk, so both $A$ and $B$ are contractible, as is $A \cap B$ which is homotopic to a line segment. However, $S^2$ is not contractible since $H_2(S^2) = \mathbb{Z}$.

Problem 2. In this problem each circle $S^1$ is identified with the unit circle in $\mathbb{R}^2 = \mathbb{C}$. Let $X = S^1 \times S^1$ be a torus. Let $p \in S^1$ and consider two subspaces $\Gamma_1 = S^1 \times \{p\}$ and $\Gamma_2 = \{p\} \times S^1$ of $X$. Consider two maps $f_1 : S^1 \to \Gamma_1$ and $f_2 : S^1 \to \Gamma_2$ given by
\( f_1(e^{i\theta}) = e^{i6\theta} \) and \( f_2(e^{i\theta}) = e^{i7\theta} \). Let \( D \) be the standard 2-dimensional disk. Compute the fundamental group and singular homology (with integer coefficients) of the space \( Y := X \cup_{f_1 \cup f_2} (D \cup D) \) is the space obtained by gluing two copies of \( D \) to \( X \) according to \( f_1 \) and \( f_2 \).

Proof.

**Problem 3.** Let \( G \) be a finitely generated abelian group. Find a finite-dimensional path-connected topological space \( X_G \) with \( \pi_1(X_G) = G \).

Proof. By the fundamental theorem for finitely generated abelian groups, we know that we can write \( G = \bigoplus_{i=1}^{k} \mathbb{Z} \oplus \bigoplus_{j=1}^{l} \mathbb{Z}_{m_j} \) for integers \( k, l, \) and \( m_j \). Since \( \pi_1(X \times Y) = \pi_1(X) \oplus \pi_1(Y) \), we can build our space \( X_G \) by finding topological spaces \( X_i \) with \( \pi_1(X_i) = \mathbb{Z} \) and \( \pi_1(X_{m_j}) = \mathbb{Z}_{m_j} \) for each integer \( m_j \) and taking their product. Now \( S^1 \) has the property that \( \pi_1(S^1) = \mathbb{Z} \), so it remains to find a topological space with \( \pi_1(X_{m_j}) = \mathbb{Z}_{m_j} \). However, it is a fact that for a CW-complex \( Y \), \( \pi_1(Y) \) is simply the free group generated by the number of 1-cells of \( Y \) subject to the relations imposed by the attaching maps of the 2-cells. Hence for an integer \( m_j \), if we let \( X_{m_j} \) be \( S^1 \) with a 2-cell attached to \( S^1 \) a total of \( m_j \) times, we will have that \( \pi_1(Y) = \langle x \rangle / x^{m_j} \), which is \( \mathbb{Z}_{m_j} \).

**Problem 4.** Given an integer \( p > 1 \) and integers \( l \) relatively prime to \( p \) define the lens space \( L_{l/p} \) to be the orbit space of the unit sphere \( S^3 \subseteq \mathbb{C}^2 \) under the action of the group \( \mathbb{Z}_p \) generated by the rotation

\[
\rho(z_1, z_2) = (e^{2\pi i/p}z_1, e^{2\pi i/p}z_2)
\]

Construct a CW structure on \( L_{l/p} \) and compute its cellular homology with coefficients in \( \mathbb{Z}, \mathbb{Z}_p, \) and \( \mathbb{Z}_q \) where \( q \) is prime. (By a CW structure on a space \( X \) we mean a CW complex whose underlying topological space is homeomorphic to \( X \).)

Proof. To construct a CW-structure on \( L := L_{l/p} \), we first construct a CW-structure on \( S^3 \) and then pass to the orbit space \( L \). For a given complex number \( z \in \mathbb{C} \), we write \( z = re^{i\theta} \). For \( r \) ranging from 0 to \( (p - 1) \), we create \( p \) 0-cells given by:

\[
e_r^0 = \{(z_0, 0) | \arg(z_0) = \frac{2\pi r}{p}\}
\]

We next add \( p \) 1-cells to our space attached via homeomorphisms on the boundaries. We enumerate the \( p \) 1-cells by:

\[
e_r^1 = \{(z_0, 0) | \frac{2\pi r}{p} < \arg(z_0) < \frac{2\pi (r + 1)}{p}\}
\]

Likewise, we attach \( p \) 2-cells via homeomorphisms on their boundaries. We enumerate the \( p \) 2-cells by:

\[
e_r^2 = \{(z_0, z_1) | \arg(z_1) = \frac{2\pi r}{p}\}
\]
Finally, we add p 3-cells by homeomorphisms on their boundaries. The 3-cells are listed as follows:

$$e_3^j = \{(z_0, z_1) | \frac{2\pi r}{p} < \arg(z_1) < \frac{2\pi(r + 1)}{p}\}$$

Note that the cell structure we have defined covers all of $S^3$ and therefore gives a CW-structure to the space. As $\mathbb{Z}_p$ cyclically permutes these $p$th roots of unity, we see that for each dimension $0 \leq i \leq 3$, $\sim$ relates all of the cells $e_i^j$. Therefore when we consider the orbit space $L$ we obtain a space with one cell in each dimension $0 \leq i \leq 3$.

We now proceed to describe the maps $d_i$ as they relate to $S^3$ as this will aid us in our computation of the homology groups of $L$. At each stage of our construction of the cell structure on $S^3$, we attached our cells along their boundaries by homeomorphisms; hence the $\Delta_{ij}$ coefficients which arise in the cellular boundary formula will always be plus or minus one. We begin by computing $d_1(e_1^j)$ for $0 \leq i \leq p - 1$. For each $e_1^j$, we attached a single 1-cell between $e_0^j$ and $e_0^{j+1}$ homeomorphically along the boundary (these subscripts are always to be read modulo $p$). Using the cellular boundary formula, we compute $d_1(e_1^j) = \sum_{j=0}^{p-1} \Delta_{ij} e_0^j$.

Note that the $\Delta_{ij}$ is the degree of the map $q \circ \varphi_i$, where $q$ is the quotient map from $(S^3)^{(0)}$ to $S_0^3$, and $\varphi_i$ is the attaching map of $e_i^1$. If $j \neq i$ or $i + 1$, then we did not attach a 1-cell and $\Delta_{ij} = 0$. If $j = i$ or $i + 1$, then $\varphi_i$ is a homeomorphism and thus $\Delta_{ij} = 1$ or -1, depending on orientation. Thus $d_1(e_1^j) = e_0^i - e_0^{i+1}$.

We next compute $d_2(e_1^j) = \sum_{j=0}^{p-1} \Delta_{ij} e_1^j$ for $S^3$. To compute this quantity, we must compute $\Delta_{ij} = \deg(q \circ \varphi_i)$, where $\varphi_i$ is again the attaching map of $e_i^2$ and $q$ is the collapse of the 2-skeleton onto the 1-cell $S_0^1$. Now for each $e_i^2$, we attached homeomorphically along the boundary and the boundary of each $e_i^2$ covers all 1-cells. This makes $\Delta_{ij} = 1$ for all $e_i^j$ and we have that $d_2(e_1^j) = \sum_{j=0}^{p-1} e_1^j$.

Finally, we compute $d_3(e_1^j) = \sum_{j=0}^{p-1} \Delta_{ij} e_2^j$ for $S^3$ in a similar manner to how we computed $d_1$. Note that in our cell structure for $S^3$ we only attached a three cell between adjacent two cells (again, adjacent being thought of modulo $p$). This makes $\Delta_{ij} = 0$ for $j \neq i$ or $i + 1$, where $\Delta_{ii} = 1$ and $\Delta_{i(i+1)} = -1$ as our boundaries were attached homeomorphically and with respect to orientation. Hence $d_3(e_1^j) = e_2^j - e_2^{j+1}$.

In summary, we have created a cells structure on $S^3$ having $p$ cells in each dimension. As the images of a point $(z_0, z_1)$ under the map $h$ correspond to the action of $\mathbb{Z}_p$ on the arguments of $z_0$ and $z_1$, we found that all $p$ of the cells in each dimension became identified in the orbit space; hence we obtained a CW-structure on $L$ having one cell in each dimension 0 through 3. Finally, we computed the boundary maps $d_i$ on the cell structure of $S^3$ in order to simplify the calculation of homology groups.
The boundary maps of $S^3$ were given by:

\[
\begin{align*}
d_1(e^1_i) &= e^0_i - e^0_{i+1} \\
d_2(e^2_i) &= \sum_{j=0}^{p-1} e^2_j \\
d_3(e^3_i) &= e^2_i - e^2_{i+1}.
\end{align*}
\]

We first compute the homology of $L = L_{4/p}$ with integer coefficients. For this, we consider the cellular chain complex:

\[
0 \xrightarrow{d_4} \mathbb{Z} \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0
\]

Both $d_4$ and $d_0$ must be the zero map as they are maps either to or from the trivial group. Additionally, we know that we must have $H_0(L) = \mathbb{Z}$ as $L$ is a connected space. Since $H_0(L) = \frac{\ker(d_0)}{\text{Im}(d_1)}$ and $d_0$ is the zero map, we must have $\mathbb{Z} = \frac{\mathbb{Z}}{\text{Im}(d_1)}$, so we must have that $\text{Im}(d_1) = 0$ and $d_1$ is the zero map. We next determine the map $d_2$ by seeing where $d_2$ sends the generator for $\mathbb{Z}, e^2_0$. We compute $d_2(e^2_0)$ using the cellular boundary formula to see that $d_2(e^2_0) = \sum_\beta \Delta_{\alpha\beta} e^1_\beta$. As $L$ has only one 1-cell, this formula simplifies to $d_2(e^2_0) = \Delta_{\alpha\beta} e^1_\beta$. Our computation of $\Delta_{\alpha\beta}$ is also simplified by the fact that $L$ has only one 1-cell since this makes $L^{(1)} = S^3_0$, and the collapsing map $q$ does not collapse anything. To determine the action of the attaching map $\varphi$, we compare with the $d_2$ map of $S^3$ and see how this map changes in the orbit space.

In $S^3$, we saw that $d_2(e^2_0) = \sum_{j=0}^{p-1} e^1_j$. As all of the $e^1_j$ are associated in the orbit space, we see that $d_2(e^2_0) = pe^1_j$ when we pass to the orbit space ($e^2_0$ attaches once to each of the $p$ elements of the equivalence class). Hence $d_2$ is just multiplication by $p$, and we see that $\text{Im}(d_2) = p\mathbb{Z}$. Since $H_1(L) = \frac{\ker(d_1)}{\text{Im}(d_2)}$, we have that $H_1(L) = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$. As multiplication by $p$ is injective, we see that $\ker(d_2) = 0$. Knowing that $H_2(L) = \frac{\ker(d_3)}{\text{Im}(d_2)}$ and $\ker(d_2) = 0$, we conclude that $H_2(L) = 0$. To complete this calculation, we must compute $d_3$ as $H_3(L) = \frac{\ker(d_3)}{\text{Im}(d_2)}$. However, as $d_4$ is the zero map, we have that $H_3(L) = \ker(d_3)$. Now $d_3(e^3_0) = \Delta_{\alpha\beta} e^2_\beta$ since $L$ has only one 2-cell. As was the case when we computed $d_2$, we know that $L^{(2)} = S^2$, and so we do not need to consider the collapsing map, only the attaching map $\varphi$. Now $d_3$ on $S^3$ was given by $d_3(e^3_0) = e^2_i - e^2_{i+1}$, but as $e^2_i = e^2_{i+1}$ in the orbit space, we see that $\deg(\Delta_{\alpha\beta}) = 0$ in $L$ as the element $e^2_i$ appears with its inverse in the attaching map. Thus $d_3$ is also the zero map and $H_3(L) = \ker(d_3) = \mathbb{Z}$. For a CW-complex, we know that $H_i^{CW}(X) = 0$ whenever $X$ has no $i$-cells. As $L$ only has cells in dimensions 0 through 3, we conclude that $H_i(L) = H_i^{CW}(L) = 0$ for $i > 3$. In summary, the homology groups for $L$ with
integer coefficients are:

\[ H_0(L) = H_3(L) = \mathbb{Z} \]
\[ H_1(L) = \mathbb{Z}_p \]
\[ H_i(L) = 0 \quad \forall i \neq 0, 1, 3 \]

Finally, we compute the homology of the space \( L \) with \( \mathbb{Z}_p \) coefficients. For this, we consider the following cellular chain complex:

\[
0 \overset{d_4}{\longrightarrow} \mathbb{Z}_p \overset{d_3}{\longrightarrow} \mathbb{Z}_p \overset{d_2}{\longrightarrow} \mathbb{Z}_p \overset{d_1}{\longrightarrow} \mathbb{Z}_p \overset{d_0}{\longrightarrow} 0
\]

Now all of the maps \( d_i \) for \( 0 \leq i \leq 4 \) are identical to the maps that we computed when using \( \mathbb{Z} \) coefficients; That is, the maps \( d_4, d_3, d_1, \) and \( d_0 \) are all zero maps and \( d_2 \) is multiplication by \( p \). However, multiplication by \( p \) read modulo \( p \) is the zero map, so when we compute homology using \( \mathbb{Z}_p \) coefficients we find that all of the \( d_i \) maps are zero. This means that \( H_i(L; \mathbb{Z}_p) = \ker(d_i) = \ker(d_{i+1}) = \mathbb{Z}_p \) whenever \( i \leq 3 \) and \( H_i(L; \mathbb{Z}_p) = 0 \) for \( i > 3 \).

Finally, we mention the homology groups with \( \mathbb{Z}_q \) coefficients for \( q \) a prime number. Here, it matters if \( q \) and \( p \) are coprime or not. If \( q \) is coprime to \( p \), we will have that multiplication by \( p \) is injective, and the groups will be the same as for integer coefficients (replacing \( \mathbb{Z} \) by \( \mathbb{Z}_q \)). If \( q \) is NOT coprime to \( p \) the \( q \) must divide \( p \) since \( q \) is prime, and multiplication by \( p \) will be the trivial map, and the homology groups will agree with \( \mathbb{Z}_p \) coefficients (except with \( \mathbb{Z}_q \) instead of \( \mathbb{Z}_p \)).

**Problem 5.** Show that if a topological space \( X \) is a union of contractible sets then all cup products of positive dimension vanish.

**Proof.** For this problem, I think we are to assume that the wording is meant to be that if \( X \) is a union of two contractible sets then this property holds. Otherwise, this problem is incorrect as the torus, a compact surface, admits a finite open cover; thus we can write the torus as a finite union of contractible sets, but cup products are definitely non-trivial on the torus. We therefore assume that we can write \( X = A \cup B \) where each of \( A \) and \( B \) is contractible. From the long exact sequence for the pair in cohomology, we have:

\[
\ldots \to H^{i-1}(A; R) \to H^i(X, A; R) \overset{j^*}{\to} H^i(X; R) \to H^i(A; R) \to \ldots
\]

Since \( H^k(A; R) = 0 \) for all \( k \) since \( A \) is contractible, we see that \( j^* \) induces an isomorphism between \( H^i(X, A; R) \) and \( H^i(X; R) \) for all \( i > 0 \), where \( j^* \) is the inclusion map. By an analogous argument, we can conclude that the inclusion map also induces an isomorphism between \( H^i(X, B; R) \) and \( H^i(X; R) \) when \( i > 0 \).

Next, we use the relative version of cup products to demonstrate that all cup products between \( H^i(X, A; R) \) and \( H^k(X, B; R) \) are trivial. For \( \alpha \in H^i(X, A; R) \) and \( \beta \in H^k(X, B; R) \), we know that \( \alpha \cup \beta \in H^{i+k}(X, A \cup B; R) \). However, \( X = A \cup B \) by assumption and so \( H^{i+k}(X, A \cup B; R) = H^{i+k}(X, X; R) = 0 \). Hence all cup products between \( H^i(X, A; R) \) and \( H^k(X, B; R) \) are trivial. However, the inclusion map \( j^* \)
induces an isomorphism between $H^i(X; R)$ and $H^i(X, A; R)$ for $i > 0$ and between $H^k(X, B; R)$ and $H^k(X; R)$ when $k > 0$. By the naturality of the cup product, we conclude that all cup products between $H^i(X; R)$ and $H^k(X, R)$ are trivial whenever $i, k \neq 0$.

**Problem 6.** Show that $H^n_c(X \times \mathbb{R}; \mathbb{Z}) = H^{n-1}_c(X; \mathbb{Z})$ for all $n$.

**Proof.** I am not sure how to do this problem, but I think there is a version of Meyer-Vietoris for homology groups with compact support which gives the answer fairly readily.