Complete five out of the following 6 problems. Each problem is worth 20 points. If you choose to work on all six problems, circle the problems that you wish to have graded: 1 2 3 4 5 6.

1. (a) Outline the derivation of Adams-Bashforth methods for the numerical solution of the initial value problem: $y' = f(x, y), y(x_0) = y_0$.
   (b) Derive the Adams-Bashforth formula
   \[
   y_{i+1} = y_i + h \left[ -\frac{1}{2} f(x_{i-1}, y_{i-1}) + \frac{3}{2} f(x_i, y_i) \right]. \tag{1}
   \]
   (c) Analyze the method (1). To be specific, find the local truncation error, prove convergence and find the order of convergence.

2. Let $x_0 < x_1 < \cdots < x_n$ be given real numbers. Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the values at the nodes and $f'(x_0)$ and $f'(x_n)$ be the values of the derivative at the end points of a smooth function $f(x)$.
   (a) Derive a formula for the algebraic polynomial of the lowest degree that interpolates this data.
   (b) Derive an error estimate for the interpolating polynomial assuming that the function $f(x)$ has as many derivatives as needed for your analysis.

3. The reverse Lax-Friedrichs scheme for the advection equation $u_t + cu_x = f(x, t)$ is given by
   \[
   \frac{1}{2} \left( U_j^{n+1} + U_{j-1}^{n+1} \right) - \frac{U_j^n}{c} + \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2h} = f_j^{n+1},
   \]
   where $k$ is the time step and $h$ the mesh size.
   (a) Show that the method is consistent if $\lambda = ck/h$ is constant.
   (b) What is the order of accuracy in this case?
   (c) For what values of $\lambda$ is the method stable?

4. Consider conservative finite difference methods for scalar nonlinear conservation laws $u_t + f(u)_x = 0$.
   (a) What is the definition of a monotone method, an $l_1$-contracting method and a TVD method? Prove that the Lax-Friedrichs method is both $l_1$-contracting and TVD under a suitable condition. What is the stability condition?
   (b) Prove that any monotone method is $l_1$-contracting.

5. Consider the $2 \times 2$ matrix $A$ and the $2n \times 2n$ matrix $B$ given by
   \[
   A = \begin{bmatrix} 1 & \alpha \\ -\alpha & 1 \end{bmatrix}, \quad B = \begin{bmatrix} I_n & S_n \\ S_n^T & I_n \end{bmatrix}. \tag{2}
   \]
(a) Under what conditions on $\alpha$ will Jacobi iteration converge for $A$?
(b) Under what conditions on $S_n$ will Jacobi iteration converge for $B$?
(Hint: Use the singular value decomposition of $S_n$.)

6. (a) Prove that if $A$ has this property (unit row diagonally dominant):

$$a_{ii} = 1 > \sum_{j=1, j \neq i}^{n} |a_{ij}| \quad (1 \leq i \leq n)$$

then Richardson iteration is successful.

(b) Prove that if $A$ has this property (unit column diagonally dominant):

$$a_{jj} = 1 > \sum_{i=1, i \neq j}^{n} |a_{ij}| \quad (1 \leq j \leq n)$$

then Richardson iteration is successful.