August 2014 Qualifying Exam

Differential Topology: 751 - 761

Thursday, August 21, 2014

Do two 751 problems and two 761 problems. All carry equal marks. If you think a problem is badly stated, state precisely what you think is intended, but do not interpret a question so that it becomes trivial.

Note: throughout the exam, if you are given a space $X$ that is homeomorphic to a cell complex, and you are also given a subspace $Y \subset X$ that is homeomorphic to a cell complex, then you may assume that $Y$ is a subcomplex of $X$.

Throughout the exam, $S^n$ will be the n-sphere.

751 Problems

1. (a) Let $X$ be a compact connected subset of $S^3$ homeomorphic to a 1-dimensional cell complex. Prove that $H_1(S^3 - X)$ is free abelian of the same rank as $H_1(X)$ and that $H_n(S^3 - X) = 0$ for $n > 1$.

   (b) Let $X \subset S^3$ be homeomorphic to the disjoint union of two circles, and let $Y$ be the disjoint union of two disks. Build a space $Z$ by attaching $Y$ to $S^3$ by identifying $\partial Y$ and $X$ via a homeomorphism. Compute the homology groups of $Z$.

2. Let $M$ be a compact subset of $S^3$ homeomorphic to a 3-manifold with nonempty boundary. Show that $H_1(M; \mathbb{Z})$ has no torsion.

3. Let $X$ be a surface of genus 2 and $Y$ a torus with one boundary component (i.e., a torus from which an open disk has been removed). Let $W$ be a nonseparating circle in $X$. Let $Z$ be the space obtained by attaching $Y$ to $X$ by identifying $W$ and $\partial Y$ via a homeomorphism.

   (a) Compute the fundamental group and all the homology groups of $Z$.

   (b) Show that $Z$ retracts onto the wedge of two circles. Does $Z$ deformation retract onto the wedge of two circles? Rigorously justify your answer.

   (c) Show that $Z$ has a 3-fold irregular covering space.
761 Problems

Recall that a subset $S \subset \mathbb{R}^N$ is an embedded submanifold of dimension $k$ provided that for every point $p \in S$ there is an open ball $B_{\epsilon}(p) \subset \mathbb{R}^N$ and a smooth immersion from an open set $U_{\epsilon} \subset \mathbb{R}^k$

$$\Psi: U_{\epsilon} \to \Psi(U_{\epsilon}) = B_{\epsilon}(p) \cap S \subset \mathbb{R}^N$$

such that $\Psi$ is a homeomorphism onto its image.

4. Let $F: \mathbb{R}^N \to \mathbb{R}^{N-k}$ be a smooth map. Let $p \in \mathbb{R}^{N-k}$ be a regular value of $F$. Use the Inverse Function Theorem to prove that

$$S := F^{-1}(p) := \{ x \in \mathbb{R}^N \mid F(x) = p \}$$

$S$ is an embedded $k$ dimensional submanifold of $\mathbb{R}^N$.

5. Prove that the tangent bundle to $S^3$ is trivial

$$TS^3 \cong S^3 \times \mathbb{R}^3.$$ 

by exhibiting three explicit linearly independent vector fields $X_1, X_2, X_3$ (on $S^3$).

6. Consider $S^2 \subset \mathbb{R}^3$ in the usual way. Let $p \in S^2$. Define a two form $\omega_p$ on $T_pS^2$ by the rule

$$\omega_p(v, w) := (p, v \times w)$$

where $v \times w$ is the cross product on vectors in $\mathbb{R}^3$ and $(\cdot, \cdot)$ denotes the dot product. Using stereographic coordinates (from the north pole, for example) show that $\omega_p$ varies smoothly with $p$, that is, is a smooth differential two form on $S^2$. 

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