Problem 1. Identify the vector space of all $n \times n$ real matrices with $\mathbb{R}^n$. Let $Y_n = SO(n)$ denote the topological subspace of all $n \times n$ orthogonal matrices of determinant one. Using the natural projection embedding we obtain an inclusion $Y_n \subseteq Y_{n+1}$.

1. Prove that $Y_n$ is a compact topological space for all $n \geq 1$.

2. Show that $Y_{n-1}$ is a closed subgroup of $Y_n$, and that for $n \geq 2$ the space of cosets $Y_n/Y_{n-1}$ is homeomorphic to $S^{n-1}$.

3. Using induction and part (b), show that $Y_n$ is connected for all values of $n \geq 1$.

Proof. This is more or less a point set topology question and would most likely not be asked on a current qual. However, for a proof of all of these facts, see Hatcher, page 292.

Problem 2. Let $\mathbb{RP}^n$ denote the space of unoriented lines through the origin in $\mathbb{R}^{n+1}$. Does there exist a continuous map $f : \mathbb{RP}^n \to \mathbb{R}^{n+1} - 0$ such that for all $x \in \mathbb{RP}^n$, $f(x)$ is orthogonal to $x$? Justify your answer.

Proof. No. If $f : \mathbb{RP}^n \to \mathbb{R}^{n+1} - 0$, then we can think of $f : \mathbb{RP}^n \to S^n$ since $\mathbb{R}^{n+1} - 0$ deformation retracts onto $S^n$ via collapsing all (non-trivial) vectors to the unit vector pointing in the same direction. Likewise, as the dot product measures angles between vectors and the angle between $x$ and $f(x)$ is the same as the angle between $x$ and $f(x)/\|f(x)\|$, we see that the dot product will be preserved when we switch to this viewpoint. Similarly, for $f$ to be continuous on $\mathbb{RP}^n$ we must have that $f(x) = f(-x)$ when we think of these vectors in $\mathbb{R}^{n+1}$. This means that we can think of $f$ as a map from $S^n \to S^n$ with the property that $f(x) = f(-x)$ for all $x \in S^n$. This means that $f$ is an even map, and therefore the degree of $f$ must be even. Suppose for contradiction that $f$ is such that $f(x)$ is orthogonal to $x$ for all $x \in S^n$. In particular, this gives us that $f(x) \neq x$ for all $x \in S^n$ since $\langle x, x \rangle = \|x\|^2 = 1$. Since $f(x) \neq x$ for all $x \in S^n$, we know that $f$ is homotopic to the antipodal map $a$ via the homotopy $H : X \times Y \to X$ given by $H(x, t) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$. This homotopy is continuous since the line determined by $-x$ and $f(x)$ never passes through the origin. However, $\deg(a) = (-1)^{n+1}$ is either 1 or -1. As homotopic maps have equal degrees, we see that $\deg(f) = \pm 1$. This is a contradiction as neither of these numbers is even. \qed
Problem 3. Let $\tilde{X} \to X$ be a connected $n$-sheeted covering space of a compact connected surface without boundary. Show that $\tilde{X}$ is a compact surface without boundary and that $\chi(\tilde{X}) = n\chi(X)$. Use this to explicitly determine all connected surfaces (up to homeomorphism) which are double covers of the Klein bottle.

Proof. See August 1996, problem 1; note that all surfaces are (by the classification) homeomorphic to one of $S^2$, $T^n$, or $P^n$, each of which is a CW-complex, so it suffices to prove this for CW-complexes.

Problem 4. Suppose that $m$ and $n$ are integers with $n > m > 0$. Let $M^m$ and $N^n$ denote compact, connected manifolds without boundary of dimensions $m$ and $n$, respectively. Prove that they are not homotopy equivalent.

Proof. It is a fact that homotopy equivalent spaces have $H_i(X;G) \cong H_i(Y;G)$ for all $i$ and all coefficient groups $G$; hence if $M$ and $N$ are homotopy equivalent, they will have the same homology groups over $\mathbb{Z}_2$, where all manifolds are orientable. Now for compact connected manifolds without boundary which are $\mathbb{R}$-orientable, we know that $H_d(X;R) \cong R$ for $d$ the dimension of $X$ and we know that $H_i(X;R) = 0$ for all $i > d$. Hence $H_m(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $H_i(M;\mathbb{Z}_2) = 0$ for all $i > m$. In particular, $H_n(M;\mathbb{Z}_2) = 0$ since $n > m$. However, the same fact gives us that $H_n(N;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Since 0 and $\mathbb{Z}_2$ are not isomorphic, we see that $M$ and $N$ are not homotopy equivalent.

Problem 5. Let $X = \mathbb{C}P^n$ denote complex projective space of dimension $2n$.

1. Prove that $X$ is a compact, connected $2n$-dimensional manifold.

2. Show in detail that $X$ can be given a CW-complex structure with one cell in every even dimension $i = 0, 2, \ldots, 2n$.

3. Calculate the cohomology ring $H^*(X;\mathbb{Z})$.


Problem 6. Let $M$ denote a connected, non-orientable, compact 3-manifold without boundary. Prove that its fundamental group must be an infinite group.