Problem 1. Let $\Delta \subseteq S^n \times S^n$ be the diagonal subspace. Prove that the projection map $p : S^n \times S^n \setminus \Delta \to S^n$ given by $(x, y) \mapsto x$ is a homotopy equivalence.

Proof. See January 2004, number 1.

Problem 2. Denote $O_{n+1,2} = \{(x, y) \in S^n \times S^n | x \perp y\} \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and let $p : O_{n+1,2} \to S^n$ be the projection on the first factor. Prove that there is a continuous map $\sigma : S^n \to O_{n+1,2}$ such that $p \circ \sigma = \text{id}_{S^n}$ if and only if $n$ is odd.

Proof. Suppose that such a $\sigma$ exists. Since $p \circ \sigma = \text{id}_{S^n}$ and the degree of the identity map is one, we see that $p \circ \sigma$ is a map of degree 1. Define $a : O_{n+1,2} \to O_{n+1,2}$ by $(x, y) \mapsto (x, -y)$, which is defined on $O_{n+1,2}$ since $<x, -y> = -<x, y> = -0 = 0$. Since the map $a$ is the identity map on the first component of $(x, y) \in O_{n+1,2}$, we see that $p \circ a \circ \sigma = \text{id}_{S^n}$. As such, we again have that the degree of this map must be 1. However, the degree of $a$ as a map on $O_{n+1,2}$ is simply the degree of the antipodal map on $S^n$ which is $(-1)^{n+1}$, so we must have that $1 = \deg(\text{id}_{S^n}) = \deg(a) = (-1)^{n+1}$, which happens if and only if $n$ is odd.

There is a slight hole in the above argument, which is that we must be able to define “degree” on $O_{n+1,2}$. Now the notion of degree is defined for spheres, and more generally, for orientable manifolds. As $O_{n+1,2}$ is not a sphere, we must see that $O_{n+1,2}$ is an orientable manifold. I think that $O_{n+1,2}$ is in fact homeomorphic to $S^n \times S^{n-1}$, which is an orientable manifold, for the following reason. For each $x \in S^n$, there are $S^{n-1}$ other vectors in $S^n$ which are orthogonal to $x$. We then have $S^n$ choices for the first vector in $O_{n+1,2}$ and $S^{n-1}$ choices for the second component once the first component is fixed. As $S^n \times S^{n-1}$ is an orientable manifold, this would make the proof work.

Problem 3. Let $M$ be a compact, connected, orientable surface. Let $P = \{x_0, x_1, x_2\}$ denote a collection of three distinct points on $M$. Determine $\pi_1(M - \{x_0, x_1, x_2\}, b)$, where $b \notin P$, providing careful justification at each step.

Proof. We begin by noting that we are assuming here that $M$ is in fact a surface without boundary; however, the result does not change too much if we take $M$ to be a surface with boundary, as we will discuss. By the classification of compact surfaces, we
know that $M$ is homeomorphic to one of $S^2, T^n,$ or $P^n$, where $T^n$ is the connected sum of $n$ tori and $P^n$ is the connected sum of $n$ projective planes. Now each of these spaces. It is a theorem that each of these spaces can be obtained as an identification space from a polygonal region $Q$ with all vertices mapping to a single point. Furthermore, if $M$ is one of $T^n$ or $P^n$ we know that the polygon $Q$ has a labeling scheme of the form $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}$ [in the case of the torus] or $a_1^2 a_2^2 \cdots a_n^2$.

First, suppose that $M$ is homeomorphic to $S^2$. Now we know that $S^2$ with a point removed deformation retracts onto $\mathbb{R}^2$, so it suffices to compute $\pi_1$ of $\mathbb{R}^2$ with two points removed. If $\gamma_1$ and $\gamma_2$ are two loops in $\mathbb{R}^2$ based at $b$ which enclose $x_1$ and $x_2$ (say), then we see that $\mathbb{R}^2$ with two points removed deformation retracts onto the wedge sum of $\gamma_1$ and $\gamma_2$ based at $b$, which has fundamental group $\mathbb{Z} \ast \mathbb{Z}$.

Now suppose $M$ is either $T^n$ or $P^n$, and let $Q$ denote the associated polygonal region. Let $B$ be a ball around the three points $x_0, x_1,$ and $x_2$, which we can assume to lie on the interior of $Q$. We use Van Kampen’s theorem with $U = B_ε$, where $B_ε$ is a ball surrounding $P$ with radius $ε$ less than the radius of $B$, and $V = Q - B$. Now $V$ deformation retracts onto the boundary of $Q$, which is simply the wedge sum of either $2n$ or $n$ circles, respectively for $T^n$ and $P^n$. In the case of $T^n$ we therefore have $\pi_1(V) = \langle a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle$ and in the case of $P^n$ we have $\pi_1(V) = \langle a_1 a_2 \cdots a_n \rangle$.

Now $\pi_1(U)$ in both cases deformation retracts onto the wedge sum of three circles, say $\pi_1(U) = \langle γ_0, γ_1, γ_2 \rangle$ for $γ_i$ a loop based at $b$ surrounding the removed point $x_i$. Now $U \cup V$ is homotopic to a circle, so suppose that $\pi_1(U \cup V) = \langle c \rangle$. When we include the loop $c$ inside $U$, we see that $c \sim γ_0 γ_1 γ_2$. When we include the loop $c$ inside $V$, we see that $c$ is homotopic to the loop given by the labeling scheme of the edges of $Q$. In either case, we see by Van Kampen’s theorem that $\pi_1(M - \{x_0, x_1, x_2\}, b)$ is the free group on either $(2n) + 3$ or $n + 3$ generators (for $T^n$ and $P^n$ respectively) modulo the relationship that the loop given by the labeling scheme on $Q$ is homotopic to $γ_0 γ_1 γ_2$.

If we solve this relation for say, $γ_0$, we see that one of our generators in either case is superfluous; hence $\pi_1(T^n - \{x_0, x_1, x_2\}, b)$ is the free group on $2(n + 1)$ generators while $\pi_1(P^n - \{x_0, x_1, x_2\}, b)$ is the free group on $n + 2$ generators.

Now if $M$ is a manifold with boundary, we would simply need to be able to compute $\pi_1(M - x)$ for $x$ a point in $M$ easily in order to apply Van Kampen’s theorem with the same result; thus the early comments that, theoretically, boundary does not interfere with this method of computation too seriously.

\[\square\]

**Problem 4.** Show that $\mathbb{CP}^{2n}$ admits no orientation reversing homotopy equivalence.

**Proof.** See January 2004, number 3, part (d).

\[\square\]

**Problem 5.** Let $(X, x_0)$ and $(Y, y_0)$ denote two pointed spaces with non-trivial rational homology in positive dimensions. Prove that there is no retraction $X \times Y \rightarrow X \vee Y$, where

$$X \vee Y = \{(x, y) \in X \times Y | x = x_0 \text{ or } y = y_0\}$$

**Proof.** Suppose for contradiction that such a retraction $r$ existed. Then if $i : X \vee Y \rightarrow X \times Y$ is the inclusion map, we must have $r \circ i = id_{X \vee Y}$. Since $H_i(X; \mathbb{Q}) \neq 0$ for all $i$, we have in particular that $H_1(X; \mathbb{Q}) \neq 0$. As $H_1(X; \mathbb{Q}) = H_1(X; \mathbb{Z}) \otimes \mathbb{Q}$ by the UCT for homology, we see that $H_1(X; \mathbb{Z}) \neq 0$. In fact, we know that $\mathbb{Z} \subseteq H_1(X; \mathbb{Q})$ since
\( Z_n \otimes \mathbb{Q} = 0 \) for all \( n \). The same is true for \( Y \), or course, so we see that \( \mathbb{Z} \subseteq H_1(Y; \mathbb{Z}) \). In particular, we can conclude that \( \pi_1(X) \) contains a free part, as does \( \pi_1(Y) \) as \( H_1 \) is a quotient of \( \pi_1 \).

Now \( X \) and \( Y \) are pointed spaces, so we know by Van Kampen’s theorem that \( \pi_1(X \vee Y) = \pi_1(X) \ast \pi_1(Y) \). Likewise, we know that \( \pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y) \). As each of \( \pi_1(X) \) and \( \pi_1(Y) \) contains a copy of \( \mathbb{Z} \), we see that \( \pi_1(X) \ast \pi_1(Y) \) is not abelian. However, \( \pi_1(X) \times \pi_1(Y) \) contains \( \mathbb{Z} \oplus \mathbb{Z} \) instead of \( \mathbb{Z} \ast \mathbb{Z} \); this demonstrates that \( \pi_1(X \times Y) \) is a strictly smaller group that \( \pi_1(X \vee Y) \), and hence the map \( r \circ i \) cannot possibly factor through \( \pi_1(X \times Y) \) and remain the identity map. This gives us a contradiction, and we see that no such retraction can exist.

Alternatively, one could look at the reverse induced map on cohomology rings; this also works.

**Problem 6.** Let \( f : S^{2n-1} \to S^n \) denote a continuous map, and \( H_f = D^{2n} \cup_f S^n \) the space obtained by attaching a \( 2n \)-cell to \( S^n \) using \( f \).

1. Calculate the integral homology of \( H_f \).
2. Find an \( n \) and an \( f \) such that \( H^*(H_f, \mathbb{F}_2) \) has a non-trivial product. Justify your answer.

**Proof.** First, suppose that \( n > 1 \). Then \( H_f \) is a CW-complex having one cell in dimensions \( 0, n \), and \( 2n \). We then have the following cellular chain map: \( 0 \to \mathbb{Z} \to 0 \to \ldots \to \mathbb{Z} \to 0 \to \ldots \to \mathbb{Z} \to 0 \to 0 \)

If we compute \( H_i(H_f) = \ker(d_i)/\text{Im}(d_{i+1}) \), we see that as every map is the trivial map, we have \( H_i(H_f) = \mathbb{Z} \) if \( i = 0, n, \) or \( 2n \) and \( H_i(H_f) = 0 \) otherwise. Now suppose that \( n = 1 \) and \( f \) is a map of degree \( d \). We then have the following cellular chain complex:

\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z} & \xrightarrow{d_2} & \mathbb{Z} & \xrightarrow{d_1} & \mathbb{Z} & \xrightarrow{d_0} & 0
\end{array}
\]

Now \( d_3 \) and \( d_4 \) are both the trivial map as they are maps either from or to the trivial group. Similarly, as \( H_f \) is a connected space we see that \( H_0(H_f) = \mathbb{Z} = \mathbb{Z}/\text{Im}(d_1) \), and we conclude that \( d_1 \) must also be the trivial map. We must therefore determine \( \text{Im}(d_2) \). As \( D^2 \) is attached via the map \( f \) which has degree \( d \), we see that \( d_2(e^2) = de^1 \), for \( e^2 \) and \( e^1 \) the generators of \( C_2(H_f) \) and \( C_1(H_f) \), respectively. Hence \( H_1(H_f) = \ker(d_1)/\text{Im}(d_2) = \mathbb{Z}/d\mathbb{Z} \cong \mathbb{Z}_d \) and as multiplication by \( d \) is injective over \( \mathbb{Z} \), we have that \( H_2(H_f) = \ker(d_2) = 0 \).

Let \( n = 1 \) and let \( f \) be a map of degree \( 2 \). If we work over \( \mathbb{Z}_2 = \mathbb{F}_2 \), we have the following cellular chain complex:

\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z}_2 & \xrightarrow{d_2} & \mathbb{Z}_2 & \xrightarrow{d_1} & \mathbb{Z}_2 & \xrightarrow{d_0} & 0
\end{array}
\]

As multiplication by \( 2 \) over \( \mathbb{Z}_2 \) is the trivial map, we see that \( H_i(H_f; \mathbb{Z}_2) = H^i(H_f; \mathbb{Z}_2) = \mathbb{Z}_2 \) for \( i = 0, 1, \) or \( 2 \) and \( H_i(H_f; \mathbb{Z}_2) = H^i(H_f; \mathbb{Z}_2) = 0 \) otherwise; in fact, \( H_f = \mathbb{RP}^2 \), which is orientable over \( \mathbb{Z}_2 \). As the cup product pairing is non-singular for orientable manifolds, we see that if \( \alpha \in H^1(H_f; \mathbb{Z}_2) \) is a generator, then there must exist a non-trivial \( \beta \in H^1(H_f; \mathbb{Z}_2) \) so that \( \alpha \cup \beta \) generates \( H^2(H_f; \mathbb{Z}_2) \). As \( \alpha \) is the only non-trivial element of \( H^1(H_f; \mathbb{Z}_2) \), we see that \( \alpha \cup \alpha \neq 0 \). This gives an example of \( H_f \) for which we have a non-trivial cup product. \( \square \)