Problem 1. Let $\Delta \subseteq S^n \times S^n$ denote the diagonal subspace. Show that the projection $f : (S^n \times S^n \setminus \Delta) \to S^n$ given by $(x,y) \mapsto x$ is a homotopy equivalence.

Proof. To show that $f$ is a homotopy equivalence, we must exhibit a map $g : S^n \to (S^n \times S^n \setminus \Delta)$ so that $f \circ g \sim \text{id}_{S^n}$ and $g \circ f \sim \text{id}_{S^n \times S^n \setminus \Delta}$. Let $g : S^n \to (S^n \times S^n \setminus \Delta)$ send $x$ to $(x, -x)$. Then $f \circ g(x) = f(x, -x) = x$, so $f \circ g = \text{id}_{S^n}$. We must therefore show that $g \circ f$ and the identity map on $S^n \times S^n \setminus \Delta$ are homotopic; that is, there is a map $H : (S^n \times S^n \setminus \Delta) \times I \to (S^n \times S^n \setminus \Delta)$ with the property that $H(x,y,0) = g \circ f(x,y) = (x, -x)$ and that $H(x,y,1) = (x,y)$. Consider the map $H$ given by $H(x,y,t) = (x, \frac{ty-(1-t)x}{\|ty-(1-t)x\|^2})$. This map is continuous as the line through $y$ and $-x$ never passes through the origin since $y \neq x$ for all $x \in S^n$. We see that $H$ has the desired properties; that is, $H(x,y,0) = (x, -x)$ and $H(x,y,1) = (x,y)$, so we have the homotopy we desire, and we have shown that the spaces are homotopy equivalent. 

Problem 2. Let $X = \mathbb{C}P^n$ denote complex projective space of dimension $2n$.

1. Prove that $X$ is compact, connected $2n$-dimensional manifold.
2. Show in detail that $X$ can be given a CW-complex structure with one cell in every even dimension $i = 0, 2, \ldots, 2n$.
3. Calculate the cohomology ring $H^*(X; \mathbb{Z})$.
4. Show that if $n$ is even, then $\mathbb{C}P^n$ admits no orientation reversing homotopy equivalence.

Proof. We start with the definition of $\mathbb{C}P^n$ as the set of lines through the origin in $\mathbb{C}^{n+1}$, i.e. a vector $\vec{v} \in \mathbb{C}^{n+1}$ which passes through the origin becomes associated with $\lambda \vec{v}$ for all $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Alternatively, we can represent a vector by its unit vector and obtain $\mathbb{C}P^n$ as a quotient of the unit sphere of $\mathbb{C}^{n+1}$ under the association $\vec{v} \in S^{2n+1}$ is associated with all $\lambda \vec{v}$ with $|\lambda| = 1$. As $\mathbb{C}P^n$ can be obtained as a quotient of $S^{2n+1}$ which is both compact and connected, we see that $\mathbb{C}P^n$ is both compact and connected. Likewise, we are obtaining $\mathbb{C}P^n$ as an orbit space of a Hausdorff manifold under a fixed point free action ($\lambda \vec{v} = \vec{v}$ if and only if $\lambda = 1$). It is a theorem then that $\mathbb{C}P^n$ is also a manifold of the same (real) dimension as $S^{2n+1}$, which is $2n$. 

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Next, we inductively place a CW-structure on \( CP^n \) so that it has exactly one cell in each even dimension 0, 2, \ldots, 2n. When \( n = 1 \), we have \( CP^1 \cong S^2 \), which we know has a cell structure with exactly one cell in dimensions 0 and 2, so our base case is established. So assume that we have given a CW-structure to \( CP^{n-1} \) with exactly one cell in each dimension 0, 2, \ldots, 2(n − 1). As we proved in part (1), \( CP^n \) is obtained as an orbit space of \( S^{2n+1} \) under the action of the unit circle in \( C \). We can alternatively obtain \( CP^n \) from \( D^{2n} \) in the following way. Think of \( D^{2n} \subseteq S^{2n+1} \) as the set of vectors of \( S^{2n+1} \) with their last co-ordinate real and non-negative. We can pick as a representative for each vector \( \tilde{v} \in S^{2n+1} \) a representative for \( CP^n \) which lies in \( D^{2n} \), with the only possible ambiguity lying on the boundary of \( D^{2n} \) which is a \( S^{2n-1} \). here, on \( \partial D^{2n} \), we still have the relationship \( \tilde{v} \sim \lambda \tilde{v} \) for \( |\lambda| = 1 \). However, induction tells us that this is the space \( CP^{n-1} \), so we see in this way that \( CP^n \) can be obtained by attaching a \( 2n \)-cell to \( CP^{n-1} \), giving \( CP^n \) a CW-structure having precisely one cell in each even dimension 0, 2, \ldots, 2n.

We first claim that \( CP^n \) is an orientable manifold for all \( n \). Now \( \pi_1(CP^1) = \pi_1(S^1) = 0 \). We know that for CW-complexes \( Y \) that \( \pi_1(Y) = \pi_1(Y^{(2)}) \), so we see that \( \pi_1(CP^n) = \pi_1(CP^1) = 0 \) for all \( n \). As such, we see that for all \( n, \pi_1(CP^n) \) has no subgroup of index 2. It is therefore a fact that \( X \) has no connected two-fold cover, and we have that \( CP^n \) is an orientable manifold for all \( n \). We now show that \( H^*(CP^n) = \mathbb{Z}[\alpha]/(\alpha^{n+1}) \) for \( |\alpha| = 2 \). Again proceed by induction. Now \( CP^1 \cong S^2 \) which has the correct ring structure \( (\mathbb{Z}[\alpha]/(\alpha^2)) \), so our base case is established. So assume by induction that \( CP^{n-1} \) has the ring structure \( \mathbb{Z}[\alpha]/(\alpha^n) \). This tells us that \( \alpha^{n-1} \in H^{2n-2}(CP^n; \mathbb{Z}) \) is a generator. As \( CP^n \) is an orientable manifold, we know that the cup product pairing is non-singular; that is, there exists \( \beta \in H^2(CP^n; \mathbb{Z}) \) such that \( \alpha^{n-1} \cup \beta \in H^{2n}(CP^n; \mathbb{Z}) \) is a generator. As \( H^2(CP^n; \mathbb{Z}) = \mathbb{Z} \), we see that we can write \( \beta = k\alpha \) for some \( k \in \mathbb{Z} \). So \( \alpha^{n-1} \cup \beta = \alpha^{n-1} \cup k\alpha = k\alpha^n \) is a generator of \( H^{2n}(CP^n; \mathbb{Z}) = \mathbb{Z} \). Since \( k, \alpha^n \) are both integers and \( k\alpha^n \) generates \( \mathbb{Z} \), we see that \( \alpha^n = \pm 1 \), hence \( k \) and \( \alpha^n \) must both be \( \pm 1 \). Either way, we see that \( \alpha^n = \pm 1 \) and thus \( \alpha^n \) generates \( H^{2n}(CP^n; \mathbb{Z}) \), as desired.

Finally, we show that if \( n \) is even then there is no orientation reversal homotopy equivalence of \( CP^n \). This means that we will show that there is no \( f : CP^n \to CP^n \) such that \( f_*([CP^n]) = -[CP^n] \); i.e., no \( f \) such that \( f_* : H_{2n}(CP^n) \to H_{2n}(CP^n) \) sending 1 to -1. It therefore suffices to show that there is no \( f^* : H^{2n}(CP^n) \to H^{2n}(CP^n) \) as these two maps are dual. Now any homotopy equivalence of \( CP^n \) must induce an isomorphism on all cohomology groups of \( CP^n \). Let \( \alpha \in H^2(CP^n; \mathbb{Z}) \) be such that \( H^*(CP^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}) \). As \( \alpha \) is a generator of \( H^2(CP^n; \mathbb{Z}) = \mathbb{Z} \), we see that any homotopy \( f \) of \( CP^n \) must either send \( \alpha \) to itself or to \(-\alpha \). If \( f^*(\alpha) = \alpha \), then \( f^*(\alpha^n) = \alpha^n = -\alpha^n \), so any such \( f \) would NOT induce an orientation reversing map on \( H^{2n}(CP^n) \). Alternatively, if \( f^*(\alpha) = -\alpha \), we have \( f^*(\alpha^n) = f^*(\alpha^n) = (-\alpha^n) = (-1)^n\alpha^n = \alpha^n \) since \( n \) is even. In either case, we see that \( f^*(\alpha^n) \neq -\alpha^n \), so there is no self homotopy equivalence of \( CP^n \) which reverses orientation when \( n \) is odd.

\[ \square \]

**Problem 3.** Let \( r_d : Z \to Z_d \) be the mod \( d \) reduction homomorphism, and \( r_{dp} : H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{Z}_d) \) be the induced homomorphism on cohomology. For \( c \in H^2(X; \mathbb{Z}_2) \), define \( \beta(c) \in H^4(X; \mathbb{Z}_4) \) as follows:
1. Find \( \tilde{c} \in H^2(X; \mathbb{Z}) \) such that \( r_2(\tilde{c}) = c \).

2. Define \( \beta(c) = r_4(\tilde{c} \cup \tilde{c}) \).

Show that if \( X \) is a compact, simply connected 4-manifold without boundary that \( \beta(c) \) is well defined, i.e. that \( \tilde{c} \) exists and that \( \beta(c) \) is independent of the choice of \( c \).

**Proof.** I just want to start out with the warning that my proof of this is very tedious.

We begin with some computations as to the homology groups of \( X \). Since \( \pi_1(X) = 0 \), we see that \( \pi_1(X) \) has no subgroup of index two and is therefore an orientable surface. From this we know that \( H_1(X) = \mathbb{Z} = H_0(X) \) and we know that \( H_2(X) \) has no torsion. By Poincare Duality, we know that \( H^i(X; \mathbb{Z}) \cong H_{n-i}(X; \mathbb{Z}) = 0 \). By the UCT for cohomology, we know that \( H^3(X; \mathbb{Z}) = H_3(X)/T_3 \oplus T_2 \), where \( T_i \) represents the torsion subgroup of \( H_i(X) \). Since \( H^3(X) \) is trivial, we can conclude that \( H_3(X) \) has no free part. From this, we know that \( H_3(X) = 0 \) since it has no free part and no torsion. Finally, we conclude from the UCT that \( H_2(X) \) has no torsion; we write \( H_2(X) = \bigoplus_i \mathbb{Z} \). Again by the UCT, we know that \( H^2(X) \cong H_2(X)/T_2 \oplus T_1 \). Since \( T_1 = 0 \) and \( T_2 = 0 \), we know that \( H^2(X; \mathbb{Z}) \cong H_2(X; \mathbb{Z}) \cong \bigoplus_i \mathbb{Z} \). We now wish to compute \( H^2(X; \mathbb{Z}_2) \); we again use the UCT. Now \( H^2(X; \mathbb{Z}_2) = \text{Hom}(H_2(X), \mathbb{Z}_2) \oplus \text{Ext}(H_1(X), \mathbb{Z}_2) \). Since \( H_1(X) = 0 \), we see that \( \text{Ext}(H_1(X), \mathbb{Z}_2) = 0 \) and \( H^2(X; \mathbb{Z}_2) \cong \text{Hom}(H_2(X), \mathbb{Z}_2) \cong \bigoplus_i \mathbb{Z}_2 \). This shows that the map from \( H_2(X_2; \mathbb{Z}_2) \to H^2(X_2; \mathbb{Z}_2) \) is just the map \( r_2 \); i.e. the map which takes the tuple \((x_1, \ldots, x_i) \mapsto (x_1 \equiv \text{mod } 2, \ldots, x_i \equiv \text{mod } 2)) \). We therefore see by the UCT that we have an isomorphism from \( H_2(X_2; \mathbb{Z}_2) \cong H^2(X_2; \mathbb{Z}_2) \) and therefore that such a \( \tilde{c} \) exists for all \( c \in H^2(X; \mathbb{Z}_2) \).

We now must show that \( \beta(c) \) does not depend on the chosen lift \( \tilde{c} \), i.e. if \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are two lifts of \( c \), then \( r_4(\tilde{c}_1 \cup \tilde{c}_1) = r_4(\tilde{c}_2 \cup \tilde{c}_2) \). Now if \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are both lifts of \( c \), say \( \tilde{c}_1 = (x_1, \ldots, x_l) \) and \( \tilde{c}_2 = (y_1, \ldots, y_l) \), then we know that these two points are equivalent mod 2; that is, \( x_1 \equiv y_1 \text{ mod } 2 \), \ldots, \( x_l \equiv y_l \text{ mod } 2 \). We can therefore rewrite the point \( \tilde{c}_2 = (x_1 + 2n_1, \ldots, x_l + 2n_l) \) where each \( n_i \in \mathbb{Z} \). We wish to show that \( r_4(\tilde{c}_1 \cup \tilde{c}_1) = r_4(\tilde{c}_2 \cup \tilde{c}_2) \). To do this, we will show that each coordinate of \( \tilde{c}_2 \cup \tilde{c}_2 \) is off from the corresponding coordinate of \( \tilde{c}_1 \cup \tilde{c}_1 \) by a multiple of 4. To do this, we compute \( \tilde{c}_2 \cup \tilde{c}_2 \):

\[
\tilde{c}_2 \cup \tilde{c}_2 = (x_1 + 2n_1, \ldots, x_l + 2n_l) \cup (x_1 + 2n_1, \ldots, x_l + 2n_l) \\
= ((x_1 + 2n_1) \cup (x_1 + 2n_1), \ldots, (x_l + 2n_l) \cup (x_l + 2n_l)) \\
= (x_1^2 + 4n_1 x_1 + 4n_1^2, \ldots, x_l^2 + 4n_l x_l + 4n_l^2) \\
= (x_1^2 + 4n_1(x_1 + n_1), \ldots, x_l^2 + 4n_l(x_l + n_l))
\]

From this computation, we see that \( \tilde{c}_2 \cup \tilde{c}_2 \) differs from \( \tilde{c}_1 \cup \tilde{c}_1 = (x_1^2, \ldots, x_l^2) \) by a multiple of 4, and hence \( r_4(\tilde{c}_1 \cup \tilde{c}_1) = r_4(\tilde{c}_2 \cup \tilde{c}_2) \), as wanted. \( \square \)
Problem 4. Let $D$ be a standard 2-disk with the boundary $S^1$ and let $x \neq y$ be two points in the interior of $D$. Show that there is no homeomorphism $f : D \to D$ that satisfies the following:

1. $f \circ f = id_D$
2. $f|_{S^1} = id_{S^1}$
3. $f(x) = y$

Proof. Suppose for contradiction that such a homeomorphism $f$ exists. Note first that as $f(x) = y$ and $f^2 = id_D$ that $f(y) = f(f(x)) = x$. We consider the space $X = D \setminus \{x, y\}$ and the map induced on $X$ by $f$. Note that $f$ is still not the identity map on $X$ as $f$ is continuous and therefore cannot swap only the points $x$ and $y$. Now $X$ deformation retracts to the wedge sum of two circles; let $a$ and $b$ be the loops in $\pi_1(X) = \mathbb{Z} \ast \mathbb{Z}$ which generate $\pi_1(X)$. Now $f$ is not the identity map and therefore $f_* : \pi_1(X) \to \pi_1(X)$ is not the identity map. Now $f_*(a)$ must be a generator, so we must have that $f_*(a) = b$ and as $f^2 = id_D$ we have that $f^2$ must also be the identity on $X$; hence $f_*(b) = f_*(f_*(a)) = a$. Let $\gamma$ be the loop in $\pi_1(X)$ corresponding to traveling around the boundary of $D$ counterclockwise. Now $\gamma \sim ab$, yet we must have that $f_*(\gamma) = \gamma$ since $f|_{S^1} = id_{S^1}$. We therefore have that $ab \sim \gamma = f_*(\gamma) = f_*(ab) = f_*(a)f_*(b) = ba$. However, this is a contradiction as $ab$ is not homotopic to $ba$ in $X$ as $\pi_1(X)$ is not commutative; therefore no such $f$ exists. \qed