Problem 1. Let $f : S^2 \rightarrow S^2$ be a continuous map such that $||f(p) - p|| < 1$ for all $p \in S^2$. Must $f$ be onto? Prove your answer.

Proof. We prove that $f$ must be surjective by showing that $f$ does not have degree zero, as all maps which are not surjective have degree zero. Now we note that for $p \in S^2$, $||-p-p|| = ||-2p|| = 2||p|| = 2$, so it can never be the case that $f(p) = -p$. Thus $f$ is a map such that $f(p) \neq -p$ for all $p \in S^2$. From this, we can show that $f$ is homotopic to the identity map. Since $f(p) \neq -p$ for all $p \in S^2$, we see that the line between $f(p)$ and $p$ never passes through the origin. Hence $H(p, t) = (1-t)f(p) + tp$ is a map with the property that $H(p, 0) = f(p)$ and $H(p, 1) = p$. As homotopic maps have equal degrees, we see that $\deg(f) = \deg(id_{S^2}) = 1$, proving that $f$ must be surjective.

Problem 2. Is there a map $f : S^6 \rightarrow \mathbb{C}P^3$ of non-zero degree?

Proof. No. To show that no such map exists, we wish to show that for any $f : S^6 \rightarrow \mathbb{C}P^3$, we have $f_* : H_0(S^6) \rightarrow H_0(\mathbb{C}P^3)$ being the trivial map. It suffices to show that $f^* : H^0(\mathbb{C}P^3) \rightarrow H^0(S^6)$ is the trivial map as these two maps ($f_*$ and $f^*$) are dual, and the dual of the trivial map is the trivial map. For this, we use the ring structure of these two spaces. Now $H^*(S^6) = \mathbb{Z}[\alpha]/(\alpha^2)$ for $|\alpha| = 6$, and $H^*(\mathbb{C}P^3) = \mathbb{Z}[x]/(x^4)$ for $|x| = 2$. From this ring structure, we know that the generator of $H^0(\mathbb{C}P^3)$ is $x^3$. We wish to show that $f^*(x^3) = 0$. Now $f^*(x^3) = f^*(x^3)$ as the cup product is functorial. So to determine the image of $f^*(x^3)$, it suffices to determine the image in $H^2(S^6)$ of $x$ under $f^*$. Yet $H^2(S^6) = 0$, so we see that $f^*(x) = 0$, making $f^*(x^3) = f^*(x^3)$ trivial as well. Hence any map from $S^6 \rightarrow \mathbb{C}P^3$ induces the trivial map on $H^0$, and we can conclude from this that every map from $S^6$ to $\mathbb{C}P^3$ has degree zero.

Problem 3. Show that the spaces $S^2 \times \mathbb{R}P^4$ and $S^4 \times \mathbb{R}P^2$ have the same fundamental groups, i.e. that $\pi_1(S^2 \times \mathbb{R}P^4)$ and $\pi_1(S^4 \times \mathbb{R}P^2)$ are isomorphic. Are $S^2 \times \mathbb{R}P^4$ and $S^4 \times \mathbb{R}P^2$ homotopy equivalent?

Proof. To see that these two spaces have the same fundamental group, we use the fact that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. Since $\pi_1(S^n) = 0$ for all $n \geq 2$ and $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ for all $n \geq 2$, we have $\pi_1(S^2 \times \mathbb{R}P^4) \cong \pi_1(S^2) \times \pi_1(\mathbb{R}P^4) = 0 \times \mathbb{Z}_2 \cong \mathbb{Z}_2$. Similarly, we have $\pi_1(S^4 \times \mathbb{R}P^2) \cong \pi_1(S^4) \times \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_2$. However, we know that these spaces are not homotopy equivalent. For this, we compute the cohomology groups of the two spaces using Kunneth’s formula and show that they are not isomorphic.
Now $H^i(S^n) = \mathbb{Z}$ if $i = 0, n$ and $H^i(S^n) = 0$ otherwise. Similarly for $\mathbb{R}P^n$ with $n$ even, we know from the UCT that $H^i(\mathbb{R}P^n) = \mathbb{Z}_2$ when $0 < i \leq 2n$ and $i$ is even, $H^0(\mathbb{R}P^n) = \mathbb{Z}$, and $H^i(\mathbb{R}P^n) = 0$ otherwise. As all of $S^2, S^4, \mathbb{R}P^2$ and $\mathbb{R}P^4$ are CW-complexes and both $H^i(S^2)$ and $H^i(\mathbb{R}P^4)$ are finitely generated free $\mathbb{Z}$ modules for all $i$, we can use Kunneth’s formula. This tells us $H^k(S^2 \times \mathbb{R}P^4) \cong \bigoplus_{i+j=k} H^i(S^2) \otimes H^j(\mathbb{R}P^4)$ and that $H^k(S^4 \times \mathbb{R}P^2) \cong \bigoplus_{i=j=k} H^i(S^4) \otimes H^j(\mathbb{R}P^2)$. In particular, we use this formula to show that $H^4(S^2 \times \mathbb{R}P^4) \neq H^4(S^4 \times \mathbb{R}P^2)$. Since $0 \otimes G = 0$ for all $G$, we see that $H^4(S^2 \times \mathbb{R}P^4) \cong H^0(S^2) \otimes H^4(\mathbb{R}P^4) \oplus H^2(S^2) \otimes H^2(\mathbb{R}P^4)$, and using the fact that $\mathbb{Z} \otimes G \cong G$ for all abelian groups $G$, we see that $H^4(S^2 \times \mathbb{R}P^4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Similarly, we find from Kunneth’s formula that $H^4(S^4 \times \mathbb{R}P^2) \cong H^4(S^4) \otimes H^0(\mathbb{R}P^2)$. However, this gives us that $H^4(S^4 \times \mathbb{R}P^2) \cong \mathbb{Z}$. Since $\mathbb{Z}$ is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, we see that the two given spaces have different cohomology groups and therefore they cannot be homotopy equivalent spaces.

\[ \square \]

**Problem 4.** Consider the map $f : A \times B \rightarrow S^2$, where

\[
\begin{align*}
A &= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\} \\
B &= \{(x, y, z) \in \mathbb{R}^3 : (y - 1)^2 + z^2 = 1, x = 0\}
\end{align*}
\]

and

\[
f(p, q) = \frac{p - q}{|p - q|}
\]

for $p \in A$ and $q \in B$. Is the map $f$ homotopic to the constant map? Prove your answer.

**Proof.** We know that two maps between manifolds are homotopic if and only if they have the same degree. As the constant map is not surjective, it has degree 0, so to show that $f$ is not homotopic to the constant map, it suffices to show that $\deg(f) \neq 0$. To compute $\deg(f)$, we look for a point $s \in S^4$ with $f^{-1}(s)$ finite, and we then use the fact that the degree of $f$ is equal to the sum of the local degrees of $f$ near $s$. Let $p = (x_1, y_1, 0) \in A$ and let $q = (0, y_2, z_2) \in B$, so that $p - q$ can be thought of as a position vector with the same direction as $f(p - q)$. We wish to determine all possible preimages of the point $(0, 0, -1)$. Now if $f(p, q) = (0, 0, -1)$ then we must have $(x_1, y_1 - y_2, z_2) = \lambda(0, 0, -1)$ for some $\lambda \in \mathbb{R}^+$ with $\lambda \neq 0$. This gives us a series of equations:

\[
\begin{align*}
x_1 &= \lambda \cdot 0 & (1) \\
y_1 - y_2 &= \lambda \cdot 0 & (2) \\
z_2 &= -\lambda & (3)
\end{align*}
\]

From equation (1), we gather that $x_1 = 0$, and since $p \in A$ we know that we must have $x_1^2 + y_1^2 = 1$, so $y_1 = \pm 1$. Equation (2) then gives us that $y_2 = \pm 1$. If $y_2 = -1$, we can use the fact that $q \in B$ to see that $\lambda = z_2$ must satisfy $(-2)^2 + \lambda^2 = 1$,
meaning that we must have $\lambda^2 = -3$, which cannot happen. Therefore $y_2 = 1$, which tells us that $\lambda^2 = 1$ so $\lambda = \pm 1$. As $\lambda \in R^+$, we conclude that $\lambda = 1$ and the point $q = (0,1,1)$ and $p$ must be either $(0,-1,0)$ or $(0,1,0)$. We see by inspection that $f((0,-1,0),q) = (0,-2,-1) \neq \lambda(0,0,-1)$ for any $\lambda \in R^+$, so $p$ must be $(0,1,0)$. This algebraic calculation shows that the point $(0,0,-1)$ has a unique preimage, so $\deg(f)$ is exactly equal to the local degree around $(0,0,-1)$.

Now $f$ is a continuous function since $|p-q|$ is never 0 since $A$ and $B$ are disjoint sets. As $f$ is a continuous function between two manifolds, we know that there exists an open neighborhood $U$ around $(p,q)$ for which $f|_U$ is a local homeomorphism. As such, we know that the local degree of $f|_U$ is either 1 or -1. Either way, $\deg(f) = \pm 1 \neq 0$, and we see that $f$ cannot be homotopic to the constant map since the two functions have different degrees. \qed