Problem 1. Let $f : X \to Y$ be a continuous function. Define the mapping cylinder of $f$ to be the quotient space

$$Z_f = X \times I \cup Y / \mathcal{S}$$

Where $\mathcal{S}$ is the relation which identifies the points $(x, 1)$ and $f(x)$. Let $i : X \to Z_f$ be given by $i(x) = [(x, 0)]$ and $j : Y \to Z_f$ be given by $j(y) = [y]$. Identify $X$ and $Y$ with their images under $i$ and $j$ respectively, so that $i$ and $j$ become inclusion maps.

1. Show that there is a retraction $r : Z_f \to Y$ such that $r \circ i = f$.

2. Show that in fact $Y$ is a strong deformation retract of $Z_f$.

3. Use this to show that $f$ induces an integral homology equivalence between $X$ and $Y$ (i.e. an isomorphism between all the homology groups with integer coefficients) if and only if $H_* (Z_f, X; \mathbb{Z}) = 0$.

4. Use part (c) to show that if a continuous map induces an integral homology equivalence, then it induces one with coefficients in any abelian group.

Proof. For part (1), we must construct a map $r : Z_f \to Y$ so that $r \circ i = f$. Due to the nature of the construction of $Z_f$, we must define $r$ piecewise. Let $r(y) = y$ for all $y \in Y$ and define $r(x, t) = (x, 1) = f(x)$. Note that $r$ is well defined as $r(x, 1) = (x, 1) = id_f(x)$. Now $r(i(x, 0)) = r(x, 0) = f(x)$, so we see that $r \circ i = f$.

For part (2), we wish to show that $i \circ r$ is homotopic to the identity on $Y$; that is, we wish to define a homotopy $H : Z_f \times I \to Y$ between $i \circ r$ and the identity. Again, we define $H$ piecewise. For $y \in Y \subseteq Z_f$, we define $H(y, s) = y$ for all time parameters $s$ and $y \in Y$. On $X \times I \subseteq Z_f$, we define $H(x, s, t) = (x, (1 - s)t + s)$, so that $H(x, t, 0) = (x, t)$ and $H(x, t, 1) = (x, 1) = f(x) \subseteq Y$. Again note that this homotopy is well defined as the piecewise definitions agree on $X \times 1 \equiv f(X)$.

For part (3), we look at the long exact sequence that we have in homology for the pair $(Z_f, X)$:

$$\ldots \rightarrow H_1(X; \mathbb{Z}) \xrightarrow{i_*} H_1(Z_f; \mathbb{Z}) \rightarrow H_1(Z_f, X; \mathbb{Z}) \rightarrow H_0(X; \mathbb{Z}) \xrightarrow{i_*} \ldots$$

Now if $H_i(Z_f, X; \mathbb{Z}) = 0$ for all $i$ then the exactness of the sequence gives us that $H_i(X; \mathbb{Z}) \cong H_i(Z_f; \mathbb{Z})$ via the map $i_*$. By part (2), we know that $Y$ is a deformation
conclude that for all \( k \) important facts about the tensor product; it distributes over sums, \( \otimes \)

**Proof.**
We note that by the UCT for homology, we have

\[
\text{Tor}(Z; \mathbb{Z}) \cong H_i(X; \mathbb{Z}) \rightarrow H_i(Y; \mathbb{Z})
\]

which we know to be an isomorphism, is actually induced by \( f_* \). Similarly, we know that if \( f_* \) induces a homology equivalence for all \( i \), then \( \phi_i : H_i(X; \mathbb{Z}) \rightarrow H_i(Y; \mathbb{Z}) \) must be an isomorphism for all values of \( i \). This means that both maps to and from \( H_i(Z_f, X; \mathbb{Z}) \) in the long exact sequence are the trivial map: calls these maps \( \theta \) and \( \varphi \), respectively. By the first isomorphism theorem, we have that \( H_i(Z_f, X; \mathbb{Z})/\ker(\varphi) \cong \text{Im}(\varphi) \). By exactness, we know \( \ker(\varphi) = \text{Im}(\theta) = 0 \). However, \( \varphi \) is the trivial map, so we see \( H_i(Z_f, X; \mathbb{Z}) \cong 0 \) for all \( i \), proving the other direction of the if and only if statement.

For part (4), we use the naturality of the UCT for homology. Now a long exact sequence also exists for homology with any coefficient group given by:

\[
\ldots \rightarrow H_i(X; G) \rightarrow H_i(Z_f; G) \rightarrow H_i(Z_f, X; G) \rightarrow H_{i-1}(X; G) \rightarrow \ldots
\]

Suppose \( f \) induces an integral homology equivalence. Then by part (3) we have \( H_i(Z_f, X; \mathbb{Z}) = 0 \) for all \( i \). From the UCT, we have:

\[
H_i(Z_f, X; G) \cong H_i(Z_f, X; \mathbb{Z}) \otimes G \oplus \text{Tor}(H_{i-1}(Z_f, X; \mathbb{Z}), G)
\]

However, we know \( H_i(Z_f, X; \mathbb{Z}) = 0 \) for all \( i \), and this gives us that \( H_i(Z_f, X; G) = 0 \) for all coefficient groups \( G \) and for all \( i \). Hence the long exact sequence with \( G \) coefficients has 0 for every \( H_i(Z_f, X; G) \) term and therefore gives us an isomorphism between \( H_i(X; G) \) and \( H_i(Z_f; G) \), and the retraction from \( Z_f \rightarrow Y \) via \( r \) gives us the isomorphism from \( f_* : H_i(X; G) \rightarrow H_i(Y; G) \).

\[\square\]

**Problem 2.** Let \( A \subseteq \mathbb{R}^3 \) denote the union of the three coordinate axes in \( \mathbb{R}^3 \), i.e.

\[
A = \{(x, y, z) \in \mathbb{R}^3 | x = y = 0 \text{ or } y = z = 0 \text{ or } x = z = 0\}
\]

Let \( X = \mathbb{R}^3 - A \). Compute \( H_*(X; \mathbb{Z}) \).

**Proof.** See January 2010, problem 1.

\[\square\]

**Problem 3.** Let \( X \) and \( Y \) be finite CW-complexes such that \( H_*(X \times Y; \mathbb{Q}) \cong H_*(X; \mathbb{Q}) \). What can you say about \( H_*(Y; \mathbb{Q}) \)? Justify your answer in detail.

**Proof.** We note that by the UCT for homology, we have \( H_i(Z; \mathbb{Q}) \cong H_i(Z; \mathbb{Z}) \otimes \mathbb{Q} \oplus \text{Tor}(H_{i-1}(Z, \mathbb{Q})) \) for all spaces \( Z \). As \( \mathbb{Q} \) is torsion free and \( \text{Tor}(A, B) = 0 \) if either \( A \) or \( B \) is torsion free, we have that \( H_i(Z; \mathbb{Q}) \cong H_i(Z; \mathbb{Z}) \otimes \mathbb{Q} \) for all \( i \). Also, we know three important facts about the tensor product; it distributes over sums, \( \mathbb{Z} \otimes \mathbb{Q} \cong \mathbb{Q} \) and \( Z_n \otimes \mathbb{Q} = 0 \) for all \( n \). From these facts, we can conclude that for any topological space \( Z \), \( H_i(X; \mathbb{Q}) \) is a finitely generated \( \mathbb{Q} \) module, and we can apply Künneth’s formula to conclude that for all \( k \), we have the following:

\[
H_k(X \times Y; \mathbb{Q}) \cong \bigoplus_{i+j=k} H_i(X; \mathbb{Q}) \otimes \mathbb{Q} H_j(Y; \mathbb{Q})
\]

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We claim that $\tilde{H}_j(Y; \mathbb{Q}) = 0$ for all $j$. We proceed by induction. Now $H_0(\mathbb{X} \times \mathbb{Y}; \mathbb{Q}) \cong H_0(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_0(Y; \mathbb{Q})$. As $H_0(\mathbb{X} \times \mathbb{Y}; \mathbb{Q}) \cong H_0(X; \mathbb{Q})$, we see that we must have $H_0(Y; \mathbb{Q}) = \mathbb{Q}$ and hence $\tilde{H}_0(Y; \mathbb{Q}) = 0$.

Now assume inductively that we know $H_l(Y; \mathbb{Q}) = 0$ for all $l \leq j - 1$. We show that $H_j(Y; \mathbb{Q}) = 0$. By Kunneth’s formula, we have:

$$H_j(X \times Y; \mathbb{Q}) \cong H_0(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_j(X; \mathbb{Q}) \oplus \cdots \oplus H_j(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_0(Y; \mathbb{Q})$$

By induction, we know $H_l(Y; \mathbb{Q}) = 0$ for all $l \leq j - 1$. This gives us:

$$H_j(X; \mathbb{Q}) \cong H_j(X \times Y; \mathbb{Q}) = (H_0(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_j(Y; \mathbb{Q})) \oplus H_j(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_0(Y; \mathbb{Q})$$

Now $H_0(Y; \mathbb{Q}) = \mathbb{Q}$, which makes the term $H_j(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_0(Y; \mathbb{Q}) \cong H_j(X; \mathbb{Q})$, and tells us that the term $H_0(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_j(Y; \mathbb{Q}) = 0$. However, $H_0(Z; \mathbb{Q})$ is never trivial, so the only possibility here is for $H_j(Y; \mathbb{Q}) = 0$. This proves the inductive step and shows that $H_j(Y; \mathbb{Q}) = 0$ for all $j \geq 1$ and $H_0(Y; \mathbb{Q}) = \mathbb{Q}$.

**Problem 4.** If $M$ is a compact, connected, orientable manifold of odd dimension without boundary show that $\chi(M) = 0$. Is the same true for non-orientable manifolds?

**Proof.** Say $\dim(M) = n = 2k + 1$. We know then that $\chi(M) = \sum_{i=0}^{2k+1} (-1)^i \text{rk}(H_i(M))$.

By Poincaré Duality (abbreviated P.D.) we have that $H_i(M) \cong H^{n-i}(M)$. Additionally, we can apply the Universal Coefficient Theorem (abbreviated U.C.T.), which tells us that $H^{n-i}(M) \cong T_{n-i} \oplus H_{n-i}/T_{n-i}$, where $T_i$ represents the torsion of the group $H_i(M)$. Taking the ranks of both sides, we have that $\text{rk}(H^{n-i}(M)) = \text{rk}(H_{n-i}(M)).$

As the dimension of $M$ is odd, we have when $i$ is even that $n - i$ is an odd number. We therefore have $\text{rk}(H_i(M))$ appearing in the sum for the Euler characteristic with a positive sign and $\text{rk}H_{n-i}(M)$ appearing with a negative sign, and these two terms cancel. Similarly, when $i$ is an odd number, we have that $n - i$ is an even number and the terms $(-1)^\text{rk}(H_i(M))$ and $(-1)^{n-i}\text{rk}(H_{n-i}(M))$ cancel. As the sum for $\chi(M)$ contains an even number of terms, all terms of the sum cancel in pairs, making $\chi(M) = 0$.

The same result is true even when we loose the hypothesis about orientability. To see this, we must work over $\mathbb{Z}_2$ coefficients, as all manifolds are orientable here. In switching from $\mathbb{Z}$ to $\mathbb{Z}_2$, we must replace $\text{rk}(H_i(M))$ by $\dim(H_i(M; \mathbb{Z}_2))$, and we wish to conclude therefore that $\sum_{i=0}^{2k+1} (-1)^i \dim(H_i(M; \mathbb{Z}_2)) = 0$. We must also check that this sum is truly equal to the Euler characteristic, $\sum_{i=0}^{2k+1} (-1)^i \text{rk}(H_i(M))$. By the same argument as above, we know that $\dim(H_i(M; \mathbb{Z}_2)) \cong \dim(H^i(M; \mathbb{Z}_2))$, so it suffices to show $\sum_{i=0}^{2k+1} (-1)^i \dim(H^i(M; \mathbb{Z}_2))$, and we use the UCT. This tells us that $H^i(M; \mathbb{Z}_2) = \text{Ext}(H_{i-1}(M), \mathbb{Z}_2) \oplus \text{Hom}(H_i(M), \mathbb{Z}_2)$. For each $\mathbb{Z}$ summand of $H_i(M)$,
we have that $\text{Ext}(\mathbb{Z}, \mathbb{Z}_2) = 0$ as $\mathbb{Z}$ is free and $\text{Hom}(\mathbb{Z}, \mathbb{Z}_2) \cong \mathbb{Z}_2$, so we see that a $\mathbb{Z}$ summand of $H_i(M)$, which contributes one to the rank of $H_i(M)$, contributes a $\mathbb{Z}_2$ term to $H^i(M; \mathbb{Z}_2)$, which contributes one to the dimension of $H^i(M; \mathbb{Z}_2)$. It remains to be shown that torsion terms of $H_i(M)$, which contribute nothing to the rank of $H_i(M)$, contribute nothing to $\dim(H_i(M; \mathbb{Z}_2))$. Now a $\mathbb{Z}_m$ summand of $H_i(M)$ when $m$ is odd contributes nothing to either $\text{Hom}(H_i(M), \mathbb{Z}_2)$ or $\text{Ext}(H_i(M), \mathbb{Z}_2)$ when computing $H^{i+1}(M; \mathbb{Z}_2)$ as $\text{Ext}(G, \mathbb{Z}_m) \cong G/mG$ for all $G$ and $m$. Finally, for a $\mathbb{Z}_m$ summand of $H_i(M)$ when $m$ is even has $\text{Hom}(\mathbb{Z}_m, \mathbb{Z}_2) \cong \mathbb{Z}_2$, and thus contributes one to the dimension of $H^i(M; \mathbb{Z}_2)$; however, this $\mathbb{Z}_m$ will also contribute one to the dimension of $H^{i+1}(M; \mathbb{Z}_2)$ since $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}_2) \cong \mathbb{Z}_m/(2 \cdot Z_m) \cong \mathbb{Z}_2$. However, as exactly one of $i$ and $i + 1$ is even, we (say) count one too many for the dimension of $H^i(M; \mathbb{Z}_2)$, but we subtract one too many for the dimension of $H^{i+1}(M; \mathbb{Z}_2)$. Thus the contributes of a $\mathbb{Z}_m$ for $m$ even cancel in the alternating sum $\sum_{i=1}^{2k+1} (-1)^i \dim(H^i(M; \mathbb{Z}_2))$. This shows that the sums is question are in fact the same, and the same argument as with the orientable case shows that the terms $\dim(H^i(M; \mathbb{Z}_2))$ and $\dim(H^{n-i}(M; \mathbb{Z}_2))$ all cancel each other out. \qed