Problem 1. Let \( p : \tilde{X} \rightarrow X \) be a cover. Suppose that \( f, g : Y \rightarrow \tilde{X} \) are maps such that \( p \circ f \) and \( p \circ g \) are equal and assume that \( f \) and \( g \) agree at \( y_0 \in Y \). Show that if \( Y \) is connected, then \( f = g \).

Proof. This is really more of a point set problem and the proof is in Hatcher, but I’ll prove it here as well. We will show that the set of points of \( Y \) on which \( f \) and \( g \) agree is both an open and a closed set. Then as this set is non-empty (\( y_0 \) is included, for example) we know that it must be the whole space. Let \( U_\alpha \subseteq X \) be a neighborhood around \( x = p \circ f(y) = p \circ g(y) \) so that \( p^{-1}(x) \) is a disjoint union of homeomorphic open sets \( V_\alpha \). Write \( V_{\alpha,f} \) and \( V_{\alpha,g} \) for the open sets in \( \tilde{X} \) which contain \( f(y) \) and \( g(y) \), respectively. Since both \( f \) and \( g \) are continuous, there exists a neighborhood \( N \subseteq Y \) so that \( f(N) \subseteq V_{\alpha,f} \) and so that \( g(N) \subseteq V_{\alpha,g} \). If \( f(y) \neq g(y) \) then \( V_{\alpha,f} \neq V_{\alpha,g} \) so \( f \neq g \) on all of the open neighborhood \( N \); this makes the set of points on which \( f \) and \( g \) agree closed. On the other hand if \( f(y) = g(y) \), then \( V_{\alpha,f} = V_{\alpha,g} \) so \( f = g \) on \( N \). This makes the set of points on which \( f \) and \( g \) agree open. \( \square \)

Problem 2. For \( n \geq 1 \), let \( f : D^{n+1} \rightarrow \mathbb{R}^2 \) be a map such that \( f(-x) = -x \) for all \( x \in \partial D^{n+1} \). Prove that \( f^{-1}(0) \cap \partial D^{n+1} = \emptyset \).

Proof. Suppose for contradiction that \( f^{-1}(0) \cap \partial D^{n+1} = \emptyset \). As \( f|_{S^n} : S^n \rightarrow \mathbb{R}^2 - 0 \) deformation retracts onto \( S^1 \). By a slight abuse of notation, we use \( f : S^n \rightarrow S^1 \) to refer to this map. Note that \( f \) still has the property that \( f(x) = f(-x) \) for all \( x \in S^n \). When \( n > 1 \), let \( i \) be the inclusion map from \( S^1 \rightarrow S^n \), where we map \( x \in S^1 \) to \( (x, 0, 0, \ldots, 0) \), and let \( g \) be the composition \( i \circ f \). Then \( g \) is a map from \( S^n \) to \( S^n \) which is odd, meaning that the degree of \( g \) must also be even. However, \( g \) is clearly not surjective when \( n > 1 \), and surjective maps have degree 0, so we have our contradiction.

When \( n = 1 \), I see no problem with \( f \) being the identity (i.e. the inclusion map) on \( \partial D^2 = S^1 \) and \( f \) being an embedding of \( D^2 \) into \( \mathbb{R}^2 \). Then \( f(S^1) \) is the unit circle in \( \mathbb{R}^2 \), the odd condition of the map is satisfied, and \( f^{-1}(0) \cap S^1 = \emptyset \). \( \square \)

Problem 3. Let \( K \) be a CW-complex with 2-skeleton \( K^{(2)} \). Let \( \phi : K^{(2)} \rightarrow K \) denote the natural inclusion. Prove that the induced homomorphism \( \phi_* : H_n(K^{(2)}) \rightarrow H_n(K) \) is an isomorphism for \( n = 1 \) and surjective for \( n = 2 \).
Proof. Consider the long exact sequence in homology for the pair \((K, K(2))\):

\[
\ldots \to H_i(K(2)) \xrightarrow{\partial_*} H_i(K) \to H_i(K, K(2)) \to H_{i-1}(K(2)) \to \ldots
\]

when \(i = 1\) we have:

\[
H_2(K, K(2)) \to H_1(K(2)) \xrightarrow{\phi_*} H_1(K) \to H_1(K, K(2)) \to \ldots
\]

Now \(K(2)\) is a deformation retract of an open neighborhood containing it, so we know that \(\tilde{H}_i(K, K(2)) \cong \tilde{H}_i(K/K(2))\) via \(q_*\), where \(q : K \to K/K(2)\) is the map collapsing \(K(2)\) to a single point. Now for any CW-complex \(X\) we know that \(H_i(X) = 0\) whenever \(X\) lacks cells in dimension \(i\). As \(K/K(2)\) is a CW-complex lacking cells in dimensions one and two, we know that \(H_i(K, K(2)) \cong H_i(K/K(2)) = 0\) when \(i \in \{1, 2\}\).

The above sequence for when \(i = 1\) then becomes:

\[
0 \to H_1(K(2)) \xrightarrow{\phi_*} H_1(K) \to 0
\]

From this, we know that \(\phi_*\) is an isomorphism by the exactness of the sequence. Now when \(n = 2\) the long exact sequence yields:

\[
H_3(K, K(2)) \to H_2(K(2)) \xrightarrow{\phi_*} H_2(K) \xrightarrow{\theta} H_2(K, K(2))
\]

As stated above, we know that \(H_2(K, K(2)) = 0\), which means that the map \(\theta\) labeled above is the trivial map, and \(\ker(\theta) = H_2(K)\). By exactness, we know that \(\text{Im}(\phi_*) = \ker(\theta) = H_2(K)\), which shows that \(\phi_*\) is a surjective map when \(n = 2\). \(\square\)

**Problem 4.** Let \(\pi\) be a finitely generated abelian group. Let \(n\) be a positive integer. Construct a CW-complex \(X\) such that \(H_n(X) \cong \pi\), \(H_0(X) = \mathbb{Z}\), and \(H_m(X) = 0\) for all \(m \neq n\).

**Proof.** See January 2009, problem number 2. \(\square\)

**Problem 5.** If \(M\) is a manifold with boundary, then the **double** of \(M\) is defined by identifying two copies of \(M\) along their boundaries by the identity map. Let \(M = D^2 \bigcup \bigcup_i D_\epsilon(x_i)\), where \(\{D_\epsilon(x_i)\}\) are mutually disjoint open disks of radius \(\epsilon\) in the interior of \(D^2\) centered at \(\{x_i\}\). Let \(W\) be the double of \(M\). Determine the fundamental group and Euler characteristic of \(W\).

**Proof.** I must first note that I am going to assume \(D^2\) includes its boundary circle \(S^1\). I will also state the result [without proof, since I do not not think this is the meant interpretation] for the disk without boundary.

We begin by thinking of \(D^2\) as a hemisphere of \(S^2\); to obtain the double of \(M\), we first identify the two boundary circles of the two copies of \(M\) so that we are looking at \(S^2\) with \(2n\) open disks removed. Identifying these \(2n\) balls in pairs along their boundaries via the identity map is equivalent to adding \(n\) handles to \(S^2\). From this, we see that \(W\) is homeomorphic to the orientable surface of genus \(n\), denoted \(M_n\). We therefore know that \(M_n\) can be given a CW-structure having one 0-cell, \(2n\) 1-cells, and
one 2-cell. As $\chi(W) = \chi(M_n) = \sum (-1)^i c_i(M_n)$, where $c_i$ is the number of cells if $M_n$ in dimension $i$, we see that $\chi(W) = 2 - 2n$. Similarly, we know that the orientable surface of genus $n$ can be obtained as an identification space of a polygonal region $P$ having $4n$ sides according to the labeling scheme $a_1b_1a_1^{-1}b_1^{-1}\ldots a_nb_na_n^{-1}b_n^{-1}$. As all vertices map to a single point in this polygonal region, we know that $\pi_1(W) = \langle a_1, b_1, \ldots, a_n, b_n \rangle$.

As for the space where we are NOT using $D^2$ with its boundary circle, we get that $\pi_1(W)$ is the free group on $(n(n+1))/2$ letters and that $\chi(W) = 2 - (n(n+1))/2$.

**Problem 6.** Let $T$ be the two torus and $K$ the Klein bottle. Compute the cohomology ring $H^*(T \times K; \mathbb{Z}_2)$.

**Proof.** For this, we reference a different problem for $H^*(K; \mathbb{Z}_2)$. Now we know that for any ring $R$, the cohomology ring of $T$ is $\Lambda_R[\alpha_1, \alpha_2]$, where $\alpha_1, \alpha_2$ are generators in $H^1(T; R) = R \oplus R$. Now $\Lambda_R[\alpha_1, \alpha_2]$ represents the exterior algebra on 2 generators, meaning that $\alpha_1^2 = \alpha_2^2 = 0$ and such that $\alpha_1\alpha_2 = -\alpha_2\alpha_1$. Now computing $H^*(K; \mathbb{Z}_2)$ is a qualifying exam problem from January 2007 (problem 5), where we saw that $H^*(K; \mathbb{Z}_2) \cong \mathbb{Z}[\beta, \gamma]/(\beta\gamma = \gamma\beta, \gamma^3, \beta^3)$. Since $H^i(T; \mathbb{Z}_2)$ is a finitely generated $\mathbb{Z}_2$-module for all $i$ and both $T$ and $K$ are CW-complexes, we can use Kunneth’s formula to determine $H^*(T \times K; \mathbb{Z}_2) \cong H^*(T; \mathbb{Z}_2) \otimes H^*(K; \mathbb{Z}_2)$. 

**Problem 7.** Let $M^{2n+1}$ be a compact, connected $(2n+1)$-manifold which is possibly non-orientable. Show that the Euler characteristic of $M$ is zero.