Problem 4. Let $A$ and $B$ be real numbers. Show that there is a constant $C$ independent of $A$ and $B$, and $N$ so that
\[ \left| \int_{-N}^{+N} \left[ e^{i(Ax+Bx^2)} - 1 \right] \frac{dx}{x} \right| \leq C. \]

Idea. Since the first term in the integrand is bounded, and $1/x$ is odd, we may hope that the integrals on $[0,N]$ and $[-N,0]$ cancel each other sufficiently well.

Solution. It is easy to check that the integral makes sense around zero.
\[
\left| \int_{-N}^{+N} \left[ e^{i(Ax+Bx^2)} - 1 \right] \frac{dx}{x} \right| = \left| \int_{0}^{N} \left[ e^{i(Ax+Bx^2)} - 1 - e^{i(A(-x)+B(-x)^2)} - 1 \right] \frac{dx}{x} \right|
\]
\[
= \left| \int_{0}^{N} \left( e^{i(Ax+Bx^2)} - e^{i(-Ax+Bx^2)} \right) \frac{dx}{x} \right| = \left| \int_{0}^{N} e^{iBx^2} 2i \sin(Ax) \frac{dx}{x} \right| = 2 \left| \int_{0}^{N} e^{iBx^2} \sin(Ax) \frac{dx}{x} \right|.
\]

If $A = 0$, the whole thing is zero, and if $A < 0$, the absolute value of the integral is the same as for $-A$. So we assume $A > 0$.

The plan is to do integration by parts, but the neighborhood of zero is problematic in this sense, so we want to cut the integral in two parts. We choose the breakpoint as $1/A$, since then the integral on $[0,1/A]$ will be bounded by a uniform constant.
\[
\left| \int_{0}^{1/A} e^{iBx^2} \sin(Ax) \frac{dx}{x} \right| \leq \int_{0}^{1/A} \sin(Ax) \frac{dx}{x} \leq \int_{0}^{1/A} A \frac{dx}{x} = 1.
\]

By substitution,
\[
\int_{1/A}^{N} e^{iBx^2} \sin(Ax) \frac{dx}{x} = \int_{1}^{AN} e^{iBx^2} x^2 \sin(x) \frac{dx}{x}.
\]

We may assume that $B \geq 0$. Let $D = \frac{B}{x} \geq 0$. Also, it suffices to estimate the real and imaginary parts separately. So it is enough to prove that
\[
\int_{1}^{K} \cos(Dx^2) \frac{\sin(x)}{x} \frac{dx}{x}, \quad \text{and} \quad \int_{1}^{K} \sin(Dx^2) \frac{\sin(x)}{x} \frac{dx}{x}
\]
are bounded independently of $D \geq 0$ and $K > 1$. 

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For a moment consider the easier case $D = 0$. Then estimate the first integral by integration by parts. 

$$
\int_1^K \frac{\sin x}{x} \, dx = \left[ -\frac{\cos x}{x} \right]_1^K - \int_1^K \frac{\cos x}{x^2} \, dx = \cos 1 - \frac{\cos K}{K} + O \left( \int_1^K \frac{1}{x^2} \, dx \right) = O(1).
$$

Couldn’t we do something similar when $D > 0$? Notice that the only thing we used for this estimate is that $\frac{1}{x}$ and the integral of $\sin x$ are bounded on $[1, \infty)$, and that $\frac{1}{x}$ is monotone. To make this clearer, we prove a lemma in a more general setting.

**Lemma.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $g : [a, b] \to \mathbb{R}$ be a differentiable function. Let $F : [a, b] \to \mathbb{R}$ be an antiderivative of $f$, i.e. $F' = f$, and assume also that $g$ is monotone and $|F(x)| \leq P$ and $|g(x)| \leq Q$ for all $x \in [a, b]$ for some constants $P, Q > 0$. Then

$$
\left| \int_a^b f(x)g(x) \, dx \right| \leq 4PQ.
$$

**Proof of Lemma.**

$$
\int_a^b f(x)g(x) \, dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x) \, dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x) \, dx.
$$

The first two terms are both less than $PQ$ in absolute value, and

$$
\left| \int_a^b F(x)g'(x) \, dx \right| \leq \int_a^b |F(x)||g'(x)| \, dx \leq P \int_a^b |g'(x)| \, dx \leq P \int_a^b g'(x) \, dx = \pm P(g(b) - g(a))
$$

which is less than $2PQ$ in absolute value, so the estimate. □

We will estimate only one of the integrals, the other case goes similarly. By a trigonometric identity we have that

$$
\int_1^K \cos Dx^2 \frac{\sin x}{x} \, dx = \int_1^K \frac{\sin(x + Dx^2) + \sin(x - Dx^2)}{2x} \, dx.
$$

The first term is the easier:

$$
\left| \int_1^K \frac{\sin(x + Dx^2)}{2x} \, dx \right| = \left| \int_1^K \frac{\sin(x + Dx^2)(1 + 2Dx)}{2x(1 + 2Dx)} \, dx \right| \leq 4 \cdot 1 \cdot \frac{1}{2} = 2
$$

by the lemma if $f(x) = \sin(x + Dx^2)(1 + 2Dx)$ and $g(x) = \frac{1}{2x(1 + 2Dx)}$.

The other case is more delicate since the same tricks leads to a denominator that may be zero. This zero occurs at $x = \frac{1}{2D}$, so away from this points we try to estimate the integral by the lemma, and close to it the trivial estimate will work very nicely. So for instance if $K$ is large, and $D$ is close to zero, we have the decomposition

$$
\int_1^K \frac{\sin(x - Dx^2)}{2x} \, dx = \int_1^{1/4D} \frac{\sin(x - Dx^2)(1 - 2Dx)}{2x(1 - 2Dx)} \, dx + \int_1^{1/4D} \frac{\sin(x - Dx^2)}{2x} \, dx + \int_1^{K} \frac{\sin(x - Dx^2)(1 - 2Dx)}{2x(1 - 2Dx)} \, dx
$$

If $K$ is not bigger than $1/D$, than we don’t have the last integral, only the first and a part of the second, or a part of the first. Also, if for certain values of $D$, $1/4D$ is not bigger than 1, so there
may be many possible cases for the limits of these integrals. However, then length of the interval of the middle one is always at most \(3/4D\), so that may be estimated trivially by \(3/4D \cdot 2D = 3/2\).

Then, we may have one or two more integrals like the first and the third. Their length does not matter. The only important thing is that \(\frac{1}{2x(1-2Dx)}\) is at most \(1\) in absolute value. This function is also monotone on these intervals therefore by the lemma both integrals are at most \(4 \cdot 1 = 4\) in absolute value. All these bounds are independent of \(K\) and \(D\), so we are done. □

Problem 8C. Prove that the improper integrals \(\int_0^\infty \sin(x^2)dx\) and \(\int_0^\infty \cos(x^2)dx\) exist, and they both equal \(\sqrt{2\pi}/4\).

Idea. Clearly we are supposed to use contour integrals somehow, but it is not straightforward how to do it since the integrands are holomorphic on the whole plane, so there is no way to use the Residue Theorem somehow. We could try to do substitution or integrate by parts to make singularities, but the problem is that we will still have sin and cos in the integrals, and as soon as we try to integrate on a curve that leaves the real axis, these two functions begin to grow exponentially in absolute value, making impossible to get estimates.

So we try to calculate the integral of \(e^{ix^2}\) instead. It is a function which is easier to handle. Although along the line \(y = -x\) this function also grows exponentially, on the line \(y = x\), this function is basically \(e^{-x^2}\), something we can integrate. (In comparison, both \(\sin(z^2)\) and \(\cos(z^2)\) grow exponentially along both the \(y = x\) and the \(y = -x\) lines.)

Solution. It suffices to prove that

\[
\int_0^\infty e^{ix^2}dx = \frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}
\]

Let \(\Gamma_R\) be the contour oriented counterclockwise which consists of the line segments \([0, R]\), the segment between 0 and \(Re^{i\pi/4}\) \((\delta_R)\), and the smaller arc of the circle of radius \(R\) centered at the origin which connects the endpoints of the segments \((\gamma_R)\). Since \(e^{ix^2}\) is holomorphic on the whole plane, we have

\[
0 = \int_{\Gamma_R} e^{iz^2}dz = \int_0^R e^{ix^2}dx + \int_{\gamma_R} e^{iz^2}dz - \int_{\delta_R} e^{iz^2}dz
\]

By substituting \(z = e^{i\pi/4}x\) we get

\[
\int_{\delta_R} e^{iz^2}dz = e^{i\pi/4} \int_0^R e^{-x^2}dx,
\]

so if we prove that \(\int_{\gamma_R} e^{iz^2}dz \to 0\) as \(R \to \infty\), then we have

\[
\int_0^\infty e^{ix^2}dx = \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right) \int_0^\infty e^{-x^2}dx
\]

Now, considering the parametrization of \(\gamma_R\) as \(\theta \to Re^{i\theta}\),

\[
\left|\int_{\gamma_R} e^{iz^2}dz\right| \leq \int_0^{\pi/4} e^{iR^2e^{2\pi\theta}R e^{i\theta}}d\theta \leq R \int_0^{\pi/4} e^{-R^2\sin 2\theta}d\theta = \frac{R}{2} \int_0^{\pi/2} e^{-R^2\sin \theta}d\theta \leq \frac{R}{2} \int_0^{\pi/2} e^{-R^2/2}d\theta \leq \frac{R}{2} \int_0^{\infty} e^{-R^2/2}d\theta = \frac{R}{2} \cdot \frac{1}{R^2/2} \to 0,
\]

\[
\int_0^\infty e^{-x^2}dx = \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right) \int_0^\infty e^{-x^2}dx
\]
as $R \to \infty$.

So the only thing left to show is $\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$, or equivalently $\int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi}$. One possibility it considering the square of the integral:

$$
\left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy
$$

Now by Fubini’s Theorem, this equals to

$$
\int_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy = \int_0^{2\pi} \int_0^{\infty} r e^{-r^2} \, dr \, d\theta = \pi.
$$