Problem 6. Let $f_n \in L^1(\mathbb{R})$ for $n = 1, 2, \ldots$, and suppose $\lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)| dx = 0$. For any $\alpha > 0$, let $E_n(\alpha) = \{x \in \mathbb{R} : |f_n(x)| > \alpha\}$. For each of the following statements, either prove the statement is true, or show that it is false by giving a counterexample.

(a) There is a set $E \subset \mathbb{R}$ with $|E| = 0$ so that if $x \notin E$, then $\lim_{n \to \infty} f_n(x) = 0$.

(b) Let $\alpha > 0$. Then $\lim_{n \to \infty} \sqrt{\alpha} |E_n(\alpha)| = 0$.

(c) $\lim_{n \to \infty} \frac{1}{n} |E_n(\frac{1}{n})| = 0$.

(d) $\lim_{n \to \infty} \frac{1}{\sqrt{n}} |E_n(\frac{1}{\sqrt{n}})| = 0$.

Solution. (a) False. Take $f_n$ to be characteristic functions of intervals of length $\frac{1}{n}$ in $[0, 1]$ so that each interval begins where the previous ended. If we reach 1, we start it again from 0. (It is true, however, that there is a subsequence of $(f_n)$ which tends to zero a.e.)

(b) True, because clearly $|E_n(\alpha)| \leq \frac{\|f_n\|_{L^1}}{\alpha}$.

(c) True. If the limit was not zero, then $|E_n(\frac{1}{n})| \geq cn$ would hold for infinitely many $n$ for some fixed $c > 0$. However, that would imply

$$\int_{\mathbb{R}} |f_n(x)| dx \geq \int_{E_n} \frac{1}{n} dx \geq c,$$

a contradiction.

(d) False. $f_n = 2^n \chi_{[0, \sqrt{n}]}$ is clearly a counterexample.