Problem 1. (i) Evaluate
\[ \int_{\Gamma} (x - y^3)dx + x^3dy, \]
where \( \Gamma \) is the unit circle in \( \mathbb{R}^2 \), with counterclockwise orientation.

(ii) Find a function \( \lambda \) such that for any closed continuously differentiable curve \( C \) (i.e. a curve parametrized by a \( C^1 \) map \( t \rightarrow \gamma(t), t \in [0,1] \), with \( \gamma(0) = \gamma(1) \)),
\[ \int_C (x - y^3)dx + x^3dy = \int_C \lambda(x, y)dy. \]

Idea. (i) Use Green’s Theorem.
(ii) Subtracting the right hand side from the left, we can see that we are looking for a vector field whose integral vanishes on all curves. Such a vector field is a conservative vector field and these vector fields are precisely the gradients of functions by the gradient theorem.

Solution. (i) By Green’s Theorem,
\[ \int_{\Gamma} (x - y^3)dx + x^3dy = \int_{B_1(0)} \left( \frac{\partial}{\partial x} x^3 - \frac{\partial}{\partial y} (x - y^3) \right) dxdy = 3 \int_{B_1(0)} (x^2 + y^2) dxdy = \]
\[ = 3 \int_0^1 \int_0^{2\pi} r^2 \cdot r d\theta dr = \frac{3}{2} \pi. \]

(ii) Find a function \( g(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R} \) so that \( \nabla g(x, y) = (x - y^3)dx + (x^3 - \lambda(x, y))dy \) for some \( \lambda \). For instance, let \( g(x, y) := \frac{x^2}{2} - y^3x \). Then \( \frac{\partial}{\partial y} g(x, y) = -3y^2x \), so \( \lambda(x, y) = x^3 + 3y^2x \) is a good choice by the preceding remarks. \( \square \)

Problem 2. (i) Let \( f \) be a continuously differentiable function on \( \mathbb{R}^2 \). Assume that at each point
\[ \frac{\partial f}{\partial x_1} > \left| \frac{\partial f}{\partial x_2} \right|. \]
Show that if \( f(x_1, x_2) = f(x_1', x_2') \), then \( |x_1' - x_1| < |x_2' - x_2| \) unless \( (x_1, x_2) = (x_1', x_2') \).

(ii) Let \( \phi \) be a map from \( \mathbb{R}^2 \) into itself defined by:
\[ \phi(x_1, x_2) = (x_1 + \sin(x_1\frac{x_1}{2} + \frac{x_2}{4}), x_2 + \cos(x_1\frac{x_1}{4} + \frac{x_2}{2})). \]
Show that \( \phi \) is a diffeomorphism from \( \mathbb{R}^2 \) onto \( \mathbb{R}^2 \), i.e. a ‘smooth’ map with smooth inverse. Evaluate the partial derivatives of \( \phi^{-1} \) at the point \((0,1)\).
HINT: Question (i) can be used to prove injectivity. For surjectivity, a possibility is to prove that \( \Phi(\mathbb{R}^2) \) is closed and open in \( \mathbb{R}^2 \).

Idea. (i) The hypothesis says that the gradient of \( f \) has angle less than \( \pi/4 \) with the vector \((1, 0)\). To relate \( f(x_1, x_2) \) and \( f(x'_1, x'_2) \) we could calculate a line integral of the gradient of \( f \) between the two points.

(ii) \( \phi \) is infinitely many times differentiable since the component functions are so - they are compositions of infinitely many times differentiable functions (polynomials, \( \sin, \cos \)). So it’s enough to prove the injectivity and surjectivity, because then the smoothness of the inverse follows from the Inverse Function Theorem.

The injectivity part is easy. For the surjectivity, in particular for the closedness of the image, there are more ways to proceed. The simplest is probably observing the \( \phi \) is close to the identity, meaning that the difference \( \phi(x) - x \) is bounded by some constant.

Another, sightly more complicated, but more general way is the following. We can to use that the Jacobian of \( \phi \) is bounded both above and below by a positive constant therefore is unable to deform the plane too much (the image of a region with finite area has approximately -up to multiplication by a constant - the same area as the original region), and also that the image of a curve by \( \phi \) can’t be much longer than the original curve.

Solution. (i) Suppose that \((x_1, x_2) \neq (x'_1, x'_2)\). Let \( \gamma \) be the segment

\[
\gamma(t) = ((1-t)x_1 + tx'_1, (1-t)x_2 + tx'_2), \quad t \in [0, 1]
\]

connecting the two points. Suppose that \(|x'_1 - x_1| > |x'_2 - x_2|\). We want to prove that \( f(x'_1, x'_2) - f(x_1, x_2) \neq 0 \). Without loss of generality we may also assume that \( x'_1 - x_1 > 0 \). Then, by the gradient theorem,

\[
f(x'_1, x'_2) - f(x_1, x_2) = \int_{\gamma} \nabla f = \int_0^1 \left( \frac{\partial f}{\partial x_1}(\gamma(t))(x'_1 - x_1) + \frac{\partial f}{\partial x_2}(\gamma(t))(x'_2 - x_2) \right) dt > 0.
\]

(ii) Injectivity: Let \( \phi = (\phi_1, \phi_2) \). Then

\[
\frac{\partial \phi_1}{\partial x_1} = 1 + \frac{1}{2} \cos \left( \frac{x_1}{2} + \frac{x_2}{4} \right) \leq \frac{1}{2}, \quad \frac{\partial \phi_1}{\partial x_2} = \frac{1}{4} \cos \left( \frac{x_1}{2} + \frac{x_2}{4} \right) \leq \frac{1}{4}
\]

\[
\frac{\partial \phi_2}{\partial x_1} = -\frac{1}{4} \sin \left( \frac{x_1}{4} + \frac{x_2}{2} \right) \leq \frac{1}{4}, \quad \frac{\partial \phi_2}{\partial x_2} = 1 - \frac{1}{2} \sin \left( \frac{x_1}{4} + \frac{x_2}{2} \right) \geq \frac{1}{2}
\]

hence both \( \phi_1 \) and \( \phi_2 \) satisfy the hypothesis of part (i), \( \phi_2 \) does it with the coordinates switched. So if \( \phi(x_1, x_2) = \phi(x'_1, x'_2) \), and \((x_1, x_2) \neq (x'_1, x'_2)\), then we have \(|x'_1 - x_1| < |x'_2 - x_2|\) and also \(|x'_1 - x_1| > |x'_2 - x_2|\), a contradiction. Hence \( \phi \) is injective.

Surjectivity: Let \( D = \text{Im}(\phi) \). The injectivity implies by the Invariance of domain (or the Inverse Function Theorem) that \( D \) is open.

Now we want to show that \( D \) is closed. For this, let \( y \in \mathbb{R}^2 \), and let \( y_1, y_2, \ldots \in D \) so that \( y_n \to y \). Let \( x_1, x_2, \ldots \) such that \( \phi(x_n) = y_n \) for all \( n \). If \( x_1, x_2, \ldots \) has a convergent subsequence, then by the continuity of \( \phi \), \( y \in D \) holds. And it has, since \(|\phi(x) - x| \leq \sqrt{2}\) for all \( x \in \mathbb{R}^2 \), so for \( n \) large enough \(|x_n - y| \leq |x_n - \phi(x_n)| + |\phi(x_n) - y| < 2\).
By the Inverse Function Theorem, $D(\phi^{-1})(\phi(x)) = [(D\phi)(x)]^{-1}$. We have $\phi(0, 0) = (0, 1)$, and
\[
D\phi(0, 0) = \begin{bmatrix} \frac{3}{2} & 0 \\ \frac{1}{4} & 1 \end{bmatrix},
\]
so
\[
D(\phi^{-1})(0, 1) = \begin{bmatrix} \frac{3}{2} & 0 \\ \frac{1}{4} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{3} & 0 \\ -\frac{1}{6} & 1 \end{bmatrix},
\]
hence $\partial x_1^{-1}(0, 1) = (\frac{2}{3}, 0)$, $\partial x_2^{-1}(0, 1) = (-\frac{1}{6}, 1)$.

\begin{proof}
Solution 2. (For the closedness of the image) Observe that the inequalities proved in (i) show that $\det(D\phi) \geq \frac{1}{2} - \frac{1}{16} = \frac{4}{16}$, and it is also easy to see that $\det(D\phi) \leq \frac{3}{4} + \frac{1}{16} = \frac{19}{16}$, where $D\phi$ is the Jacobian of $\phi$. Also, note that $D\phi$ do not increase the length of vectors very much: if $x = (x_1, x_2)$, then $|\phi(x_1)| \leq \frac{3}{2}|x_1| + \frac{1}{4}|x_2|$ and $|\phi(x_2)| \leq \frac{1}{4}|x_1| + \frac{3}{2}|x_2|$, hence
\[
|\phi(x)|^2 \leq \left(\frac{3}{2}|x_1| + \frac{1}{4}|x_2|\right)^2 + \left(\frac{1}{4}|x_1| + \frac{3}{2}|x_2|\right)^2 = \frac{37}{16}(x_1^2 + x_2^2) + \frac{3}{2}|x_1||x_2| \leq \frac{37}{16}(x_1^2 + x_2^2) + \frac{3}{4}(x_1^2 + x_2^2) = \frac{49}{16}|x|^2.
\]

Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a continuously differentiable curve. Then
\[
I(\phi(\gamma)) = \int_0^1 |(\phi(\gamma(t)))'|dt = \int_0^1 |(D\phi)(\gamma(t)) \cdot \gamma'(t)|dt \leq \int_0^1 \frac{7}{4} |\gamma'(t)|dt = \frac{7}{4} I(\gamma).
\]
Now let $E \subset \mathbb{R}^2$ be a region of finite area. Then
\[
\text{area}(\phi(E)) = \int_{\phi(E)} 1 = \int_E |\det(D\phi)| \geq \frac{3}{16} \text{area}(E),
\]
\[
\text{area}(\phi(E)) = \int_{\phi(E)} 1 = \int_E |\det(D\phi)| \leq \frac{37}{16} \text{area}(E).
\]

Again, let $y \in \mathbb{R}^2$, and let $y_1, y_2, \ldots \in D$ so that $y_n \to y$. Let $x_1, x_2, \ldots$ such that $\phi(x_n) = y_n$ for all $n$.

Assume by contradiction that $x_1, x_2, \ldots$ does not have a convergent subsequence. Then by choosing a subsequence, we may assume that the disks $B_1(x_n)$ of radius 1 around them are pairwise disjoint. Their images are pairwise disjoint regions with area at least $\frac{3}{16}$, hence there is a sequence $z_1, z_2, \ldots$ such that $z_n \in B_1(x_n)$ and $|\phi(z_n)| \to \infty$. Let $L_n$ be the segment connecting $x_n$ and $z_n$. On one hand, $\phi(L_n)$ has length at most $7/4$, on the other hand these lengths should diverge to infinity since $\phi(x_n) \to y$, but $|\phi(z_n)| \to \infty$. This contradiction shows that $D = \mathbb{R}^2$, i.e. $\phi$ is surjective.
\end{proof}

Problem 3. For $\lambda > 0$, set
\[
F(\lambda) = \int_0^1 e^{-10\lambda x^4 + \lambda x^6} dx.
\]
Prove that there exists $A$ and $C > 0$ such that $F(\lambda) = \frac{A}{\lambda^{1/4}} + E(\lambda)$, where $|E(\lambda)| \leq \frac{C}{\lambda^{1/7}}$. 

On the other hand, follows from the Taylor Remainder Formula. What is stated by the problem. (Well this is actually stronger when $\lambda$ is large.) This is $A\lambda$ in notation.

For estimating the second term in the product, we use that $e^y - 1 \leq e^y y$ for $y > 0$ which easily follows from the Taylor Remainder Formula.

$$\int_0^{\lambda^{1/4}} e^{-10x^4} (e^{\lambda^{-1/2} x^6} - 1) dx \leq \int_0^{\lambda^{1/4}} e^{-10x^4} e^{\lambda^{-1/2} x^6} \lambda^{-1/2} x^6 dx = \\lambda^{-1/2} \int_0^{\lambda^{1/4}} e^x (\lambda^{-1/2} x^2 - 10) x^6 dx \leq \lambda^{-1/2} \int_0^{\lambda^{1/4}} e^{-9x^4} x^6 dx \leq \lambda^{-1/2} \int_0^{\lambda^{1/4}} e^{-9x^4} x^6 dx = c \lambda^{-1/2}.$$  

On the other hand,  

$$\int_0^{\infty} e^{-10x^4} dx - \int_0^{\lambda^{1/4}} e^{-10x^4} dx = \int_0^{\lambda^{1/4}} e^{-10x^4} dx \leq \int_0^{\lambda^{1/4}} \frac{x^3}{(\lambda^{1/4})^2} e^{-10x^4} dx = \lambda^{-3/4} \left[ -\frac{e^{-10x^4}}{10} \right]_0^{\lambda^{1/4}} = \lambda^{-3/4} \frac{e^{-10\lambda}}{40} \leq c \lambda^{-3/4}.  

Let $A = \int_0^{\infty} e^{-10x^4} dx.$ Combining these estimates we have that  

$$F(\lambda) = \lambda^{-1/4} (A + O(\lambda^{-1/2}) + O(\lambda^{-3/4})) = A \lambda^{-1/4} + O(\lambda^{-3/4}) + O(\lambda^{-1}).$$

This is $A \lambda^{-1/4} + O(\lambda^{-3/4})$ if $\lambda \geq 1.$ And if $\lambda < 1$ we can see that it holds without any calculation, because $0 < F(\lambda) < 1,$  

$$|F(\lambda) - A \lambda^{-1/4}| \leq 1 + A \lambda^{-1/4} \leq (1 + A) \lambda^{-3/4}$$

so $F(\lambda) = A \lambda^{-1/4} + O(\lambda^{-3/4})$ holds for $\lambda < 1$ as well, by possibly increasing the constant in the $O$-notation.

So (if there are no miscalculations in the precedings) we actually proved a stronger result that what is stated by the problem. (Well this is actually stronger when $\lambda \geq 1,$ but then one can prove $F(\lambda) = A \lambda^{-1/4} + O(\lambda^{-1/2})$ for $\lambda < 1$ in the same way.) 

$\square$
Problem 4. Find a sequence of bounded measurable sets in \( \mathbb{R} \) whose characteristic functions converge weakly in \( L^2(\mathbb{R}) \) to \( \frac{1}{2} \chi \), where \( \chi \) is the characteristic function of the interval \([0, 1]\).

Does there exist a sequence of bounded measurable sets in \( \mathbb{R} \) whose characteristic functions converge weakly in \( L^2(\mathbb{R}) \) to \( 2\psi \), where \( \psi \) is the characteristic function of a set of positive measure?

Recall that a sequence \( f_n \) in \( L^2(\mathbb{R}) \) tends weakly to \( f \in L^2(\mathbb{R}) \), if and only if for every \( g \in L^2(\mathbb{R}) \), \( \int f_n g \to \int f g \).

Idea. Notice that in the definition of weak convergence one can say \( g \in C(\mathbb{R}) \) instead of \( g \in L^2(\mathbb{R}) \) since the continuous functions are dense in \( L^2 \). So then how do one choose a measurable set \( S \) such that \( \int_S g \approx \frac{1}{2} \int_0^1 g \)? Take a set of measure 1/2 which consists of many small intervals, evenly distributed in \([0, 1]\).

For the other part, choose a clever test function \( g \) for which the required convergence is impossible. For example, \( \psi \) itself seems a good try.

Solution. Let \( S_n = \left[ 0, \frac{1}{2n} \right] \cup \left[ \frac{1}{2n}, \frac{3}{2n} \right] \cup \cdots \cup \left[ \frac{2n-2}{2n}, \frac{2n-1}{2n} \right] \). If \( f_n \) denotes the characteristic function of \( S_n \), then, using the uniform continuity of continuous functions on \([0, 1]\), one easily proves that \( \int f_n g \to \int \frac{1}{2} \chi_g \) for all \( g \) continuous which implies that \( f_n \to \frac{1}{2} \chi \) weakly.

For the other part, let \( \psi \) be the characteristic function of \( A \), and assume that \( f_n \to 2\psi \), and \( f_n \) are characteristic functions of bounded sets. Then \( \int f_n \psi \to \int 2\psi \psi = 2|A| \), but on the other hand, \( \int f_n \psi \leq |A| \) which is a contradiction provided \( |A| > 0 \). \( \square \)

Problem 5. (i) Let \( f \in L^1([0, 2\pi]) \). Prove that \( \int_0^{2\pi} f(x) \cos(nx) dx \to 0 \) as \( n \to \infty \).

You are asked to give a proof, not simply to quote a theorem. By essentially the same proof, that you are not asked to repeat, one also has \( \int_0^{2\pi} f(x) \sin(nx) dx \to 0 \), as \( n \to \infty \). Prove that, for any sequence \( (\alpha_n) \) in \( \mathbb{R} \), \( \int_0^{2\pi} f(x) \cos^2(nx + \alpha_n) dx \to \frac{1}{2} \int_0^{2\pi} f(x) dx \).

(ii) Let \( (a_n) \) and \( (b_n) \) be sequences in \( \mathbb{R} \) such that on a set of positive measure in \([0, 2\pi]\), \( a_n \cos nx + b_n \sin nx \) tends pointwise to 0. Prove that \( a_n \) and \( b_n \to 0 \).

Hint: Write \( a_n \cos nx + b_n \sin nx = \rho_n \cos(nx + \alpha_n) \) and use the fact that \( \cos^2(\theta) \leq |\cos \theta| \).

Idea. For the first thing we are asked to prove, let’s consider first characteristic functions of intervals for \( f \). This case is trivial. The linear combinations of these functions are dense in \( L^1 \) because they are dense in \( C([0, 2\pi]) \), and that is dense in \( L^1 \). Then, using this, proving the general case is easy.

The second thing we have to prove in part (i) follows easily from the previous ones by using some trigonometric identities.

And for the third, we already have a hint.

Solution. (i) Let \( g = \chi_{[a,b]} \) for some \( [a,b] \subset [0, 2\pi] \). Then

\[
\int_0^{2\pi} g(x) \cos(nx) dx = \int_a^b \cos(nx) dx = \frac{1}{n} (\sin(nb) - \sin(na)) \to 0,
\]

and this holds also for the linear combinations of such functions (let’s call this set of functions \( Z \)).

Now let \( f \in L^1([0, 2\pi]) \) be arbitrary. Since \( Z \) is dense in \( L^1 \), for any \( \epsilon > 0 \) there is a \( h \in Z \) such that \( ||f - h||_{L^1} < \epsilon \). Then

\[
\left| \int_0^{2\pi} f(x) \cos(nx) dx \right| = \left| \int_0^{2\pi} h(x) \cos(nx) dx \right| + \left| \int_0^{2\pi} (f(x) - h(x)) \cos(nx) dx \right| \leq \left| \int_0^{2\pi} h(x) \cos(nx) dx \right| + \int_0^{2\pi} |f(x) - h(x)| dx = \left| \int_0^{2\pi} h(x) \cos(nx) dx \right| + \epsilon,
\]
hence
\[ \limsup_{n \to \infty} \left| \int_0^{2\pi} f(x) \cos(nx) dx \right| \leq \epsilon, \]
and by letting \( \epsilon \to 0 \) we have
\[ \int_0^{2\pi} f(x) \cos(nx) dx \to 0. \]

For the other statement, we have to prove that
\[ \int_0^{2\pi} f(x)(2 \cos^2(nx + \alpha_n) - 1) dx \]
tends to 0. Observe that
\[ 2 \cos^2(nx + \alpha_n) - 1 = \cos(2nx + 2\alpha_n) = \cos(2\alpha_n) \cos(2nx) - \sin(2nx) \sin(2\alpha_n) \]
So (1) becomes
\[ \cos(2\alpha_n) \int_0^{2\pi} f(x) \cos(2nx) dx - \sin(2\alpha_n) \int_0^{2\pi} f(x) \sin(2nx) dx \]
which tends to zero since the integrals do and the cos and sin are bounded.

(ii) We can indeed write \( a_n \cos nx + b_n \sin nx \) in the form \( \rho_n \cos(nx + \alpha_n) \), since after using the addition formula for the cos, we are left to find a solution to the following system of equations:
\[ \rho_n \cos(\alpha_n) = a_n, \quad \rho_n \sin(\alpha_n) = b_n \]
(Hint: \( \rho_n = \sqrt{a_n^2 + b_n^2} \)). It suffices to show therefore that \( \rho_n \to 0 \).

Let \( f = \chi_A \) be the characteristic function of the set \( A \) where \( a_n \cos nx + b_n \sin nx \) tends pointwise to 0. According to part (i),
\[ \int_A \cos^2(nx + \alpha_n) dx = \int_0^{2\pi} f(x) \cos^2(nx + \alpha_n) dx \to \frac{1}{2} \int_0^{2\pi} f(x) dx = \frac{|A|}{2} > 0. \]
On the other hand,
\[ \int_A \cos^2(nx + \alpha_n) dx \leq \int_A |\cos(nx + \alpha_n)| dx = \int_A \frac{|a_n \cos nx + b_n \sin nx|}{\rho_n} dx. \]
Assume first that the the sequence \( \{\rho_n\} \) is unbounded. By choosing a subsequence, we may assume that \( \rho_n \to \infty \). Then \( \frac{a_n \cos nx + b_n \sin nx}{\rho_n} \) converges pointwise to zero on \( A \), and these functions are bounded by the constant 1 function, hence by Lebesgue’s Dominated Convergence Theorem
\[ \int_A \frac{|a_n \cos nx + b_n \sin nx|}{\rho_n} dx \to 0 \]
which yields \( \frac{|A|}{2} \leq 0 \), a contradiction. Hence \( \{\rho_n\} < K \) for some \( K > 0 \) for all \( n \). But now the Dominated Convergence Theorem implies that
\[ \rho_n \int_A \cos^2(nx + \alpha_n) dx \leq \int_A |\rho_n \cos(nx + \alpha_n)| dx = \int_A |a_n \cos nx + b_n \sin nx| dx \to 0. \]
since the integrand is bounded by \( K \) which is shown by the second integral in the line. But then, we must have \( \rho_n \to 0 \).  \( \square \)
Problem 6. (i) Give an example of a sequence of functions \( f_n \in \mathcal{L}^1([0,1]) \), \( n = 1, 2, \ldots \), with the following properties:

1. \( \lim_{n \to \infty} f_n(x) = 1 \) for any \( x \in [0,1] \);
2. \( \int_0^1 |f_n(x)| \, dx = 2 \) for any \( n = 1, 2, \ldots \)

(ii) Show that if the \( f_n \) are as in part (i) then

\[
\lim_{n \to \infty} \int_0^1 |f_n(x) - 1| \, dx = 1.
\]

Idea. (i) The usual bump functions with the bumps approaching 0 do the job.

(ii) Divide the interval \([0,1]\) into two parts: that part where \( f_n \) is closer to 1 than some small \( \epsilon \), and the rest. The measure of the first set tends to 1, and the integral on it is approximately 1, so on the rest we have a function on a really small set with norm 1. Subtracting one from it does not change it’s norm much which implies that \( f_n - 1 \) has around norm 1 on the whole interval \([0,1]\).

Solution. (i) If \( x \in [2/n,1] \), let \( f_n(x) = 1 \), let \( f_n(0) = 1 \), \( f_n(1/n) = c_n \), and define \( f_n \) to be linear on \([0,1/n]\) and on \([1/n,2/n]\) so that \( f_n \) is continuous. Then for the appropriate choice \( c_n > 1 \) this sequence satisfies the two conditions.

(ii) Let \( \epsilon > 0 \) fixed. Let

\[
A_n = \{ x \in [0,1] : |f_n(x) - 1| < \epsilon \}.
\]

Pointwise convergence in a \( \sigma \)-finite measure space implies convergence in measure which shows that \( |A_n| \to 1 \) if \( n \to \infty \). Let us choose an \( n \) such that \( |A_n| < \epsilon \). Then

\[
\int_0^1 |f_n(x) - 1| \, dx = \int_{A_n} |f_n(x) - 1| \, dx + \int_{[0,1] \setminus A_n} |f_n(x) - 1| \, dx = \int_{[0,1] \setminus A_n} (|f_n(x)| + O(1)) \, dx + O(\epsilon) =
\]

\[
= \int_{[0,1] \setminus A_n} |f_n(x)| \, dx + O(\epsilon) = 2 - \int_{A_n} |f_n(x)| \, dx + O(\epsilon) =
\]

\[
= 2 - \int_{A_n} (1 + O(\epsilon)) \, dx + O(\epsilon) = 1 + O(\epsilon)
\]

hence

\[
\lim \sup_{n \to \infty} \int_0^1 |f_n(x) - 1| \, dx = O(\epsilon)
\]

but this is true for any \( \epsilon \), so \( \epsilon \to 0 \) gives

\[
\lim_{n \to \infty} \int_0^1 |f_n(x) - 1| \, dx = 1.
\]

\( \square \)

Problem 7R. Let \( W \) be the space of continuous functions \( f \) on \([0,1]\), whose distributional derivative on \((0,1)\) is an integrable function.

In one variable, this simply means that \( f(x) = f(0) + \int_0^x g(t) \, dt \), for some integrable function \( g \), and then \( f' = g \). On \( W \) one considers the norm defined by

\[
||f||_W = |f(0)| + \int_0^1 |f'(t)| \, dt.
\]
Let $\Lambda$ be the space of continuous functions on $[0, 1]$, that are Hölder continuous of order $\frac{1}{2}$ (i.e. functions $f$ such that for some constant $C > 0$, for every $x$ and $y$, $|f(x) - f(y)| \leq C|x - y|^{1/2}$). On $\Lambda$ one considers the norm

$$||f||_{\Lambda} = |f(0)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{1/2}}.$$  

Equipped with these norms $W$ and $\Lambda$ are Banach spaces. And it is immediate that the diagonal $\Delta \subset W \times \Lambda$, that is the set of all $(f, f)$ for $f \in W \cap \Lambda$, is a closed subspace of $W \times \Lambda$. You are not asked to justify the above.

(i) Show that if $f \in W$ and $f' \in L^2$ (not only in $L^1$), then $f \in \Lambda$.

(ii) For each integer $N > 0$, set $u_N(x) = \frac{1}{N} \sin(Nx)$. Find constants $A_N$ such that $A_N \to 0$ as $N \to \infty$, and for every $x$ and $y \in \mathbb{R}$

$$|u_N(x) - u_N(y)| \leq A_N|x - y|^{1/2}.$$  

(iii) Prove that there is a Hölder continuous function $f$, of a Hölder exponent $\frac{1}{2}$, defined on $[0, 1]$ (i.e. $f \in \Lambda$) whose distributional derivative on $(0, 1)$, is not an integrable function (i.e. $f \notin W$).

Even if you are not able to prove the result of question (ii), you can use it for applying the open mapping theorem to the projection of $\Delta$ on $\Lambda$, when arguing by contradiction.

Idea. (i) It seems pretty straightforward to write $f(x) - f(y)$ as the integral of $f'$. Then, use the Cauchy-Schwarz Inequality.

(ii) Again, write the left hand side as the integral of the derivative. However, a simple Cauchy-Schwarz trick does not work in this case. We have to use that the integral of cos on long intervals is still small since it’s integral is zero on every interval of length $2\pi$.

(iii) We have a lot of hints about what to do...

Solution. (i) Let $f$ be as in the problem. By the Cauchy-Schwarz Inequality,

$$|f(y) - f(x)| = \left| \int_x^y f' \cdot 1 \right| \leq \left( \int_x^y |f'|^2 \right)^{1/2} \left( \int_x^y 1 \right)^{1/2} \leq ||f'||_{L^2([0, 1])}|y - x|^{1/2}.$$  

(ii) Without loss of generality we may assume that $x < y$.

$$|u_N(y) - u_N(x)| = \left| \int_x^y \cos(Nt) dt \right| = \frac{1}{N} \left| \int_{xN}^{yN} \cos(t) dt \right| = \frac{1}{N} \left| \int_{xN}^{yN-2\pi k} \cos(t) dt \right|$$

where $k$ is the largest non-negative integer such that $yN - 2\pi k \geq xN$. Now use the Cauchy-Schwarz Inequality again.

$$\frac{1}{N} \left| \int_{xN}^{yN-2\pi k} \cos(t) dt \right| \leq \frac{1}{N} \left( \int_{xN}^{yN-2\pi k} \cos^2(t) dt \right)^{1/2} (yN - 2\pi k - xN)^{1/2} \leq \frac{1}{N} \left( \int_0^{2\pi} \cos^2(t) dt \right)^{1/2} N^{1/2} |y - x|^{1/2}.$$  

(iii) Assume by contradiction that $\Lambda \subset W$. Then $\Delta = \{(f, f) : f \in \Lambda\} \subset W \times \Lambda$. Consider the projection $p : \Delta \to \Lambda$. Note that $\Delta$, as a closed linear subspace of $W \times \Lambda$, is a Banach space. By assumption $p$ is therefore a surjective map between Banach spaces. Since $p$ is continuous, by the Open Mapping Theorem, $p$ is an open map, therefore $p^{-1}$ is a continuous linear operator as well.
Problem 8R. Let \( H \) be a Hilbert space over \( \mathbb{R} \), with scalar product denoted by \( \langle \cdot, \cdot \rangle \), and associated norm denoted by \( \| \cdot \| \).

1) Assume that \( (x_n) \) and \( (y_n) \) are sequences in \( H \) such that \( \|x_n\| \leq 1, \|y_n\| \leq 1 \) and \( \langle x_n, y_n \rangle \to 1 \) as \( n \to \infty \). Show that \( \langle x_n - y_n, x_n - y_n \rangle \to 0 \) as \( n \to \infty \).

2) Let \( T \) be a continuous linear map from \( H \) into itself.

2.1) Recall what is the definition of the adjoint operator \( T^* \).

2.2) Assume that \( T \) is self adjoint, i.e. that \( T^* = T \). And assume that, for some sequence \( x_n \in H \), with \( \|x_n\| \leq 1 \):

\[
1 = \sup_{\|x\| \leq 1} \|T(x)\| = \lim_{n \to \infty} \|T(x_n)\|.\]

Using question 1, show that \( T^2(x_n) - x_n \) tends to 0 as \( n \to \infty \).

Conclude that at least one of the 2 operators \( T + 1 \) or \( T - 1 \) is not invertible. Here 1 denotes the identity map on \( H \) (i.e. \( 1(x) = x \)).

Solution. 1) Since \( \langle x_n, y_n \rangle \leq \|x_n\|\|y_n\| \leq 1 \), and the left hand side tends to 1, we have \( \|x_n\|\|y_n\| \to 1 \) by the Sandwich Theorem. But \( \|x_n\| \leq 1, \|y_n\| \leq 1 \), therefore \( \|x_n\| \to 1 \) and \( \|y_n\| \to 1 \).

On the other hand,

\[
0 \leq \langle x_n - y_n, x_n - y_n \rangle = \|x_n\|^2 + \|y_n\|^2 - 2\langle x_n, y_n \rangle \to 0
\]

so \( \langle x_n - y_n, x_n - y_n \rangle \to 0 \) again by the Sandwich Theorem which is the same as saying that \( \langle x_n - y_n, x_n - y_n \rangle \to 0 \).

2.1) For \( x \in H \), \( T^* x \) is the unique element of \( H \) so that \( \langle x, Ty \rangle = \langle T^* x, y \rangle \) for all \( y \in H \).

2.2) \( \langle T^2(x_n), x_n \rangle = \langle T(x_n), T(x_n) \rangle \to 1 \), and the other hypotheses of 1) are clearly true. So \( T^2(x_n) - x_n \to 0 \).

If \( T + 1 \) and \( T - 1 \) were both invertible, then their product, \( T^2 - 1 \) would be invertible, too. But \( 1/2 \leq \|T\| \cdot \|x_n\| \), therefore \( \|x_n\| \geq \frac{1}{2\|T\|} \) for \( n \) large enough, and

\[
\|x_n\| = \|(T^2 - 1)^{-1}(T^2 - 1)x_n\| \leq \|(T^2 - 1)^{-1}\| \cdot \|(T^2 - 1)x_n\| \to 0,
\]

a contradiction. \( \square \)

Problem 9R. We shall use the following normalization for the Fourier transform on \( \mathbb{R}^n \). When it makes (classical) sense:

\[
\hat{f}(\xi_1, \ldots, \xi_n) = \int_{\mathbb{R}^n} f(y_1, \ldots, y_n) e^{-i(y_1 \xi_1 + \cdots + y_n \xi_n)} dy_1 \cdots dy_n.
\]

And the Fourier inversion formula is thus, when it makes sense:

\[
f(x_1, \ldots, x_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi_1, \ldots, \xi_n) e^{i(x_1 \xi_1 + \cdots + x_n \xi_n)} d\xi_1 \cdots d\xi_n
\]

1) Let \( f \) be a bounded function on \( \mathbb{R}^2 \), explain how (in the theory of tempered distributions) one defines its Fourier transform \( \hat{f} \), even if \( f \) is not integrable. What is the Fourier transform of the constant function 1?
2) Let \( \phi \) be a continuous function with compact support on \( \mathbb{R} \). Its Fourier transform is therefore a continuous function \( \hat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x)e^{-\xi x}dx \). Let \( \phi \) be the function of two variables defined by
\[
\phi(x_1, x_2) = \varphi(x_1).
\]

Find the Fourier transform of \( \phi \), in terms of \( \hat{\varphi} \).

**Problem 7C.** The \( \Gamma \) function is defined for \( \Re z > 0 \) by the convergent improper integral
\[
\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt.
\]
The object of this problem is to prove that for \( 0 < \Re(z) < 1 \),
\[
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.
\]
Either present your own proof of this identity, or complete the following outline:
- Use contour integration to show that if \( x \) is real and \( 0 < x < 1 \) then
  \[
  \int_0^{\infty} \frac{e^{-t}t^{x-1}}{1 + r} dr = \frac{\pi}{\sin(\pi x)}.
  \]
- Show that if \( 0 < x < 1 \), then \( \Gamma(x)\Gamma(1-x) = \int_0^{\infty} \left[ \int_0^{\infty} e^{-s-t} \left( \frac{t}{s} \right)^x dt \right] ds \).
- Show that if \( 0 < x < 1 \), then
  \[
  \int_0^{\infty} \left[ \int_0^{\infty} e^{-s-t} \left( \frac{t}{s} \right)^x dt \right] ds = \int_0^{\infty} \frac{e^{-x}r^{x-1}}{1 + r} dr.
  \]
- Explain why the identity \( \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \) where \( \Gamma(x) = \int_0^{\infty} e^{-xt}t^{x-1}dt \) for \( 0 < x < 1 \) implies the desired identity.

**Solution.**
1) (A more detailed description of the technique of contour integration can be found in the solution of 2011 Jan/7C.) Define the branch of \( z^{-1} \) in the region \( \mathbb{C} \setminus \{0, \infty\} \) by \( z^{-1} = e^{(x-1)\log(z)} \), where \( \log(z) \) is the branch of \( \log \) in the region \( \mathbb{C} \setminus \{0, \infty\} \) so that \( 0 < \Im(\log(z)) < 2\pi \). This way we get a branch of \( e^{\frac{z}{1-z}} \), defined on \( \mathbb{C} \setminus \{(0, \infty) \cup \{-1\}\} \) which has a simple pole at \(-1\).

Then, take the closed curves \( \gamma_{\epsilon,R} \) which consist of four parts. Two segments: the segment \([1/R,R] \) shifted up and down by \( \epsilon \), and two circular arcs with center \( 0 \), and radii approximately \( 1/R, R \) which connects the left and right endpoints of the previous segments and does not cross \([0,\infty)\). Orient the curve counterclockwise. By the Residue Theorem,
\[
\int_{\gamma_{\epsilon,R}} \frac{z^{x-1}}{1+z} dz = 2\pi i \text{Res}_{z=-1} \frac{z^{x-1}}{1+z} = 2\pi i e^{(x-1)i\pi} = -2\pi i e^{ix\pi}.
\]
On the other hand,
\[
\lim_{R \to \infty} \lim_{\epsilon \to 0} \int_{\gamma_{\epsilon,R}} \frac{z^{x-1}}{1+z} dz = \int_0^{\infty} \frac{r^{x-1}}{1+r} dr - \int_0^{\infty} \frac{r^{x-1}}{1+r} e^{(x-1)2\pi i} dr = (1 - e^{2\pi ix}) \int_0^{\infty} \frac{r^{x-1}}{1+r} dr
\]
using that while \( \epsilon \to 0 \), the functions on the segments converge uniformly to the corresponding functions on \([1/R,R]\), and that while \( R \to \infty \) the integral on the small circle and the large circle around zero tends to 0.

Hence
\[
\int_0^{\infty} \frac{r^{x-1}}{1+r} dr = \frac{2\pi i e^{ix\pi}}{1 - e^{2\pi ix}} = \frac{\pi}{\sin(\pi x)}.
\]
2) By rearranging the integrals, we get
\[
\Gamma(x)\Gamma(1-x) = \int_0^\infty e^{-t} t^{x-1} dt \int_0^\infty e^{-s} s^{-x} ds = \int_0^\infty \left[ \int_0^\infty e^{-s} \left( \frac{t}{s} \right)^x \frac{dt}{t} \right] ds.
\]
3) By substituting \( t = sr \),
\[
\int_0^\infty \left[ \int_0^\infty e^{-s} t^{x-1} \frac{dt}{t} \right] ds = \int_0^\infty \left[ \int_0^\infty e^{-s-rs} r^{x} \frac{dr}{r} \right] ds.
\]
We have a double integral, so by Petruska’s Theorem, we have to interchange them. (We are allowed to since the integrand is non-negative.) So
\[
\int_0^\infty \left[ \int_0^\infty e^{-s-rs} r^{x} \frac{dr}{r} \right] ds = \int_0^\infty \left[ \int_0^\infty e^{-s-rs} ds \right] r^{1+x} \frac{dr}{1+r}.
\]
4) Now we have
\[
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},
\]
for all \( 0 < x < 1 \). One proves easily with Morera’s Theorem that \( \Gamma(x) \) is holomorphic if \( \Re z > 0 \). Hence \( \Gamma(z)\Gamma(1-z) \) and \( \frac{\pi}{\sin(\pi z)} \) are both holomorphic functions in the connected domain \( 0 < \Re z < 1 \), agreeing on a set that has an accumulation point in the interior of the domain. Therefore they coincide. \( \square \)

**Problem 8C.** Let \( f \) be a function which is holomorphic in the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), and assume that
\[
\iint_{\mathbb{D}} |f(x + iy)|^2 dx dy = M^2 < \infty.
\]
1) If \( f(z) = \sum_{n=0}^\infty a_n z^n \), prove that \( M^2 = \pi \sum_{n=1}^\infty n|a_n|^2 \).
2) Prove that \( \sup_{0 \leq r < 1} \int_{0}^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty \).
3) Prove that there are real constants \( C, \alpha > 0 \) such that \( |f'(z)| \leq C (1 - |z|)^{-\alpha} \) for all \( z \in \mathbb{D} \), and find the smallest possible value of \( \alpha \).

**Idea.** 3) A simple use of Cauchy’s Formula shows that \( \alpha = 2 \) is a good constant. However, it turns out that the best value for \( \alpha \) is 1. This is one of the very few cases when estimating the derivative using Cauchy’s Formula is not the best choice. (At least I could not prove the sharp estimate with it.)

Instead, we are going to use the power series form of \( f(z) \) which turns out to be more efficient, maybe because we have good control over the rate of growth of the coefficients. Part 1) shows what we know of the coefficients, so let’s use it for the estimate.

For proving that \( \alpha = 1 \) is the best constant, one can think in the following way. The bigger the coefficients in absolute value (still satisfying the finiteness criterion in Part 1) of course), the sharper the estimate. \( a_n = \frac{1}{n!} \) just fails to satisfy the finiteness of the sum in Part 1); the correspondent function is \( \log \frac{1}{1-z} \) with derivative \( \frac{1}{1-z} \).

Here there are basically two ways to correct it. One is that we take a function which is just a little smaller around 1, or in other words whose derivative has a singularity at 1 of a little smaller order. This leads one to try \( f(z) = (1-z)^\epsilon \), since \( f'(z) = -\epsilon (1-z)^{\epsilon-1} \). This choice is good in the sense that \( f'(z) \) is easy to estimate, however, the coefficients are hard, because they are binomial coefficients.

The other way is to define \( f(z) \) with its coefficients. Here, ensuring that the finiteness criterion in Part 1) holds is easy, but estimating the derivative is more complicated.
We shall choose the first way.

**Solution.** 1) Write $|f(z)|^2 = f(z)\overline{f(z)}$, multiply out, and integrate term by term in polar coordinates.

2) By essentially the same argument,

$$
\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = 2\pi \sum_{n=0}^\infty r^{2n} |a_n|^2 \leq 2\pi \sum_{n=0}^\infty |a_n|^2 < \infty
$$

by part (i).

3) By the Cauchy-Schwarz Inequality (why is this version of it true?)

$$
|f'(z)| = \left| \sum_{n=1}^\infty n^n a_n z^{n-1} \right| \leq \sum_{n=1}^\infty n|a_n||z|^{n-1} = \sum_{n=1}^\infty \sqrt{n}|a_n| \cdot \sqrt{n}|z|^{n-1} \leq \left( \sum_{n=1}^\infty n|a_n|^2 \right)^{1/2} \left( \sum_{n=1}^\infty n|z|^{2n-2} \right)^{1/2} = M \left( \sum_{n=1}^\infty n|z|^{2n-2} \right)^{1/2}.
$$

On the other hand, with the notation $r = |z|$.

$$
\sum_{n=1}^\infty n|z|^{2n-2} = \sum_{n=1}^\infty n|z|^{2n-2} = \frac{1}{2r} \sum_{n=1}^\infty 2nr^{2n-1} = \frac{1}{2r} \left( \sum_{n=1}^\infty r^{2n} \right)' = \frac{1}{2r} \left( \frac{r^2}{1-r^2} \right)' = \frac{1}{(1-r^2)^2}.
$$

So

$$
|f'(z)| \leq c \frac{1}{1-|z|^2} = c \frac{1}{(1-|z|)(1+|z|)} \leq c \frac{1}{1-|z|},
$$

that shows that $\alpha = 1$ is a good constant.

For showing that this is the least good value, let $\epsilon > 0$, and consider the function $f(z) = (1-z)^\epsilon = e^{\epsilon \log(1-z)}$, where the branch of the logarithm is defined in $\mathbb{C} \setminus (-\infty, 0]$ such that $-\pi < \text{Im} \log z < \pi$, so $f(z)$ is defined on $\mathbb{C} \setminus [1, \infty)$, in particular, it is defined in $\mathbb{D}$. By the Binomial Formula,

$$
f(z) = (1-z)^\epsilon = \sum_{n=0}^\infty \binom{\epsilon}{n} (-z)^n.
$$

To prove that the finiteness criterion of Part 1) holds for this function, we need to estimate the coefficients of this power series.

**Lemma.** Let $0 < \epsilon < 1$. Then there is a constant $c_\epsilon$, independent of $n$ such that

$$
\left| \binom{\epsilon}{n} \right| \leq \frac{c_\epsilon}{n^{1+\epsilon}}.
$$

**First proof of lemma.**

$$
\left| \binom{\epsilon}{n} \right| = \frac{\epsilon}{n} (1-\epsilon) \cdot \left( 1 - \frac{\epsilon}{2} \right) \cdots \left( 1 - \frac{\epsilon}{n-1} \right).
$$

Approximating sums is always easier than products. So by taking logarithm we have

$$
\log \left| \binom{\epsilon}{n} \right| = \log \epsilon - \log n + \log (1-\epsilon) + \log \left( 1 - \frac{\epsilon}{2} \right) + \cdots + \log \left( 1 - \frac{\epsilon}{n-1} \right) \leq \log \epsilon - \log n - \epsilon - \frac{\epsilon}{2} - \cdots - \frac{\epsilon}{n} \leq \log \epsilon - \log n - \epsilon (\log n),
$$
so it follows that
\[
\left| \frac{\epsilon}{n} \right| \leq \frac{\epsilon}{n^{1+\epsilon}}.
\]

For the estimate, we used that \(\log(1 + x) \leq x\), which can be seen by considering the function \(x\) as the tangent line of \(\log(1 + x)\) at \(x = 0\), and using that \(\log(1 + x)\) is concave. We also used that \(\sum_{k=1}^{n} \frac{1}{k} \geq \log n\) that can be justified either by induction or be approximating the sum the integral of \(\frac{1}{x}\). (By the way, \(c_\epsilon = \epsilon\) is the best constant we can get because equality holds for \(n = 1\).) \(\square\)

*Second proof of lemma.* Now we use integration to approximate the sum.

Let \(g(x) = \log(1 - \frac{x}{\epsilon})\). Then
\[
|a_n| = |\epsilon| \leq e^{g(1)+g(2)+\cdots+g(n-1)}. \quad \text{Since} \quad g(x) \text{ is increasing, we have}
\]
\[
|a_n| = \epsilon \frac{1}{1-\epsilon} \leq \epsilon \log \frac{n-\epsilon}{1-\epsilon} - \log(1-\epsilon) \leq -\epsilon \log n - \log(1-\epsilon).
\]

So
\[
|a_n| \leq \epsilon \frac{1}{1-\epsilon} \log n - \log(1-\epsilon) = \frac{\epsilon}{1-\epsilon} \frac{1}{n^{1+\epsilon}},
\]
where the constant is slightly worse than in the previous solution. \(\square\)

According to the lemma,
\[
|a_n| = \left| \frac{\epsilon}{n} \right| \leq \frac{\epsilon}{n^{1+\epsilon}},
\]
so
\[
\sum_{n=1}^{\infty} \frac{n|a_n|^2}{n} \leq \epsilon^2 \sum_{n=1}^{\infty} \frac{1}{n^{1+2\epsilon}} < \infty
\]
as we claimed.

Moreover,
\[
|f'(z)| = \left| -\frac{e^{f(z)}}{1-z} \right| = \epsilon |1-z|^{\epsilon-1},
\]
which shows by choosing \(z\) as real numbers that no value less than 1 is good for \(\alpha\). \(\square\)

**Problem 9C.** Let \(D = \{z \in \mathbb{C} : |z| < 1\}\) be the open unit disk, and let \(S = \{w \in \mathbb{C} : |\text{Im } w| < \frac{\pi}{2}\}\) be a horizontal strip centered at zero of width \(\pi\).

1) Find an explicit biholomorphic mapping from \(S\) to \(D\); i.e. find a holomorphic function \(\Phi : S \rightarrow D\) which is one-to-one and onto and whose inverse is also holomorphic.

2) Let \(f : S \rightarrow S\) be a holomorphic function with \(f(0) = 0\). Show that
\[
\left| \frac{e^{f(z)}}{e^{f(z)}+1} - \frac{e^z-1}{e^z+1} \right| \leq \frac{e^z - 1}{e^z + 1}.
\]

3) Let \(f : S \rightarrow S\) be a holomorphic function with \(f(0) = 0\). What can you conclude about \(f\) if \(f'(0) = 1\)? Why?

**Idea.** 1) First apply the exponential function which brings \(S\) to a halfplane. Then apply a linear fractional transformation.

2, 3) Conjugate \(f\) with the map found in Part 1) and apply the Schwartz Lemma.
Solution. 1) It is easy to check that \( \Phi(z) = \frac{z^2 - 1}{z^2 + 1} \) is a good choice.

2) Let \( g = \Phi \circ f \circ \Phi^{-1} : \mathbb{D} \to \mathbb{D} \). By the Schwarz Lemma, \( |g(w)| \leq |w| \). If \( z = \Phi^{-1}(w) \), then this yields
\[
|\Phi(f(z))| \leq |\Phi(z)|
\]
and that is what we had to prove.

3) The Schwarz Lemma also says that \( |g'(0)| \leq 1 \), and equality holds if and only if \( g(z) = cz \) where \( |c| = 1 \). In particular \( g'(0) = 1 \) implies \( g(z) = z \). And since
\[
g'(0) = \Phi'(f(\Psi^{-1}(0))) f'(\Psi^{-1}(0))(\Psi^{-1})'(0) = \Psi'(0) f'(0)(\Psi^{-1})'(0) = f'(0),
\]
f\( (0) = 1 \) implies \( f(z) = z \). \( \square \)