In all cases, when an example is requested, you should both provide the example and proof that the object you write down actually is an example.

(1) A finite group $G$ is said to have property C if, whenever $g \in G$ and $n$ is an integer relatively prime to the order of $G$, $g$ and $g^n$ are conjugate in $G$.

(a) (3pt) Give infinitely many non-isomorphic finite groups which have property C.

(b) (3pt) Give infinitely many non-isomorphic finite groups which do not have property C.

(c) (4pt) Show that if $G$ has property C and $\rho : G \to GL_m(\mathbb{C})$ is a homomorphism, then the trace of $\rho(g)$ lies in $\mathbb{Q}$ for every $g \in G$.

(2) Let $k$ be a field. We say a polynomial $f$ in $k[x]$ is a consecutive-root polynomial if it has two roots $x_0, x_1$ (not necessarily in $k$) which satisfy $x_1 - x_0 = 1$.

(a) (2pt) Show there is no irreducible consecutive-root polynomial in $\mathbb{Q}[x]$.

(b) (3pt) Let $p$ be a prime number. Show that the polynomial $x^p - x - 1$ in $\mathbb{F}_p[x]$ is irreducible and consecutive-root.

(c) (5pt) Describe the set of irreducible monic consecutive-root polynomials in $\mathbb{F}_p[x]$ of degree at most $p$.

(3) We say a ring $R$ is von Neumann regular if, for every $a \in R$, there exists an $x \in R$ such that $a = axa$. The element $x$ is called a weak inverse of $a$. In particular, every division algebra is von Neumann regular (just take $x = 0$ if $a = 0$ and $x = a^{-1}$ otherwise.)

(a) (4pt) Give an example of a commutative von Neumann regular ring which is not a field.

(b) (2pt) Let $R$ be $M_2(\mathbb{C})$ and let $a$ be the nilpotent matrix $e_{12}$ which sends $e_1$ to 0 and $e_2$ to $e_1$. Give a weak inverse for $a$.

(c) (4pt) Prove that if $V$ is a vector space over a field $k$, the ring of endomorphisms $End_kV$ is von Neumann regular.

(4) Recall that a right module $P$ for a ring $R$ is said to be projective if, for every surjection of right $R$-modules $f : N \to P$, there is a map $g : P \to N$ such that $g$ followed by $f$ is the identity on $P$.

(a) (3pt) Prove that a free $R$-module is projective.

(b) (3pt) Prove that a right $R$-module $M$ is projective if and only if there is another right module $N$ such that $M \oplus N$ is a free right $R$-module.

(c) (4pt) In linear algebra, a “projection” is a matrix $A$ such that $A^2 = A$. More generally, if $R$ is a commutative ring, we might say that an $R$-projection is an $R$-module homomorphism $A : R^n \to R^n$ such that $A^2 = A$. For $R$ a commutative ring, prove that a finitely generated $R$-module $M$ is projective if and only if it is isomorphic to the image of some projection.
(5) Let $W_n$ be the set of $n \times n$ complex matrices $C$ such that the equation

$$AB - BA = C$$

has a solution in $n \times n$ matrices $A, B$.

(a) (2pt) Show that $W_n$ is closed under scalar multiplication and conjugation.

(b) (4pt) Show that the identity matrix is not in $W_n$.

(c) (4pt) Give a complete description of $W_2$ (i.e. a criterion for determining whether a matrix $C$ is in $W_2$ other than “look around for matrices $A$ and $B$ such that $AB - BA = C$.”)