Algebra Qualifying Exam  
August 2001

Do all 5 problems.

1. Let $G$ be a finite group of order $504 = 2^3 \cdot 3^2 \cdot 7$.
   a. Show that $G$ cannot be isomorphic to a subgroup of the alternating group $\text{Alt}_7$. (5 points)
   b. If $G$ is simple, determine the number of Sylow 3-subgroups of $G$. (5 points)

2. Let $R$ be a commutative ring with 1 and let $M$ be a maximal ideal of $R$.
   a. Show that the ring $R/M^2$ has no idempotents other than 0 and 1. (4 points)
   b. We know that $M/M^2$ is naturally an $R/M$-module. If $R$ is Noetherian, prove that this module is finitely generated. (2 points)
   c. Finally, assume that $R = K[x_1, x_2, \ldots, x_t]$ is a polynomial ring in finitely many variables over the field $K$. Prove that $\dim_K(R/M^2) < \infty$. (4 points)

3. Let $F \subseteq E$ be fields and suppose $0 \neq \alpha \in E$ with $E = F[\alpha]$. Assume that some power of $\alpha$ lies in $F$ and let $n$ be the smallest positive integer such that $\alpha^n \in F$.
   a. If $\alpha^m \in F$ with $m > 0$, show that $m$ is a multiple of $n$. (2 points)
   b. If $E$ is a separable extension of $F$, prove that the characteristic of $F$ does not divide $n$. (4 points)
   c. If every root of unity in $E$ lies in $F$, show that $|E : F| = n$. (4 points)

4. Let $A$ be a real $n \times n$ matrix. We say that $A$ is a difference of two squares if there exist real $n \times n$ matrices $B$ and $C$ with $BC = CB = 0$ and $A = B^2 - C^2$.
   a. If $A$ is a diagonal matrix, show that it is a difference of two squares. (3 points)
   b. If $A$ is a symmetric matrix that is not necessarily diagonal, again show that it is a difference of two squares. (3 points)
   c. Suppose $A$ is a difference of two squares, with corresponding matrices $B$ and $C$ as above. If $B$ has a nonzero real eigenvalue, prove that $A$ has a positive real eigenvalue. (4 points)

5. Let $K$ be a field of characteristic 0 and view the polynomial ring $V = K[x]$ as a $K$-vector space. Let $M: V \to V$ be the linear operator given by multiplication by $x$, so that $M(x^n) = x^{n+1}$ for all integers $n \geq 0$. In addition, let $D: V \to V$ be the linear operator given by differentiation with respect to $x$, so that $D(x^n) = nx^{n-1}$ for all $n \geq 0$. Let $L$ denote the set of all linear operators of the form $M^iD^j$ with $i, j \geq 0$, where $M^0 = D^0 = I$ is the identity operator on $V$.
   a. Prove that $DM - MD = I$. (3 points)
   b. Show that $L$ is a $K$-linearly independent set. (4 points)
   c. For all nonnegative integers $t$, prove that $DM^t$ is in the $K$-linear span of the set $L$. (3 points)