1. Let $G$ be a finite group and let $H \subseteq G$ be a subgroup of index $|G : H| = n$.
   a. Show that $|H : (H \cap H^g)| \leq n$ for all $g \in G$. (2 points)
   b. If $H$ is a maximal subgroup of $G$ and $H$ is abelian, show that $(H \cap H^g) \triangleleft G$ for all $g \notin H$. (3 points)
   c. Now suppose that $G$ is simple. If $H$ is abelian and $n$ is a prime, prove that $H = 1$. (5 points)

2. Let $K$ be a field and let $R$ be the subring of the polynomial ring $K[X]$ given by all polynomials with $X$-coefficient equal to 0.
   a. Prove that the elements $X^2$ and $X^3$ are irreducible but not prime in the ring $R$. (5 points)
   b. Show that $R$ is a Noetherian ring, and that the ideal $I$ of $R$ consisting of all polynomials in $R$ with constant term 0 is not principal. (5 points)

3. Recall that a field $K$ is algebraically closed if every polynomial $f \in K[X]$ splits over $K$ (is a product of linear factors in $K[X]$). Now let $F \subseteq E$ be an algebraic field extension.
   a. If every polynomial $f(X) \in F[X]$ splits over $E$, prove that $E$ is algebraically closed. (4 points)
   b. If every polynomial $f(X) \in F[X]$ has a root in $E$ and if $F$ has characteristic 0, prove that $E$ is algebraically closed. (6 points)

4. Let $V$ be a finite dimensional vector space over the field $F$. Suppose $T : V \to V$ is a linear operator and let $f(X) \in F[X]$ be its minimal polynomial.
   a. If $f(X)$ has a nonconstant polynomial factor of degree $m$, show that $V$ has a nonzero subspace $W$ of dimension $\leq m$ with $T(W) \subseteq W$. (5 points)
   b. Conversely, if $V$ has a nonzero subspace $W$ of dimension $n$ with $T(W) \subseteq W$, show that $f(X)$ has a nonconstant polynomial factor of degree $\leq n$. (5 points)

5. Let $R$ be a ring with 1 and let $V$ be a right $R$-module. Suppose that $V = X \oplus Y$ is the internal direct sum of the two nonzero submodules $X$ and $Y$.
   a. Show that 0, $X$, $Y$ and $V$ are the only $R$-submodules of $V$ if and only if $X$ and $Y$ are nonisomorphic simple $R$-modules. (6 points)
   b. If $X$ and $Y$ are nonisomorphic simple $R$-modules, prove that $\text{End}_R(V)$, the ring of $R$-endomorphisms of $V$, is isomorphic to the direct sum of two division rings. (4 points)