ANALYSIS SEP - PROBLEM SET 2
SELECTED PROBLEMS

BALAZS STRENNER

Problem 1. (2010 Aug/2) Let

\[ s_N(x) = \sum_{n=1}^{N} (-1)^n \frac{x^{3n}}{n^{2/3}}. \]

Prove that \( s_N(x) \) converges to a limit \( s(x) \) on \([0, 1]\) and that there is a constant \( C \) so that for all \( N \geq 1 \) the inequality

\[ \sup_{x \in [0,1]} |s_N(x) - s(x)| \leq CN^{-2/3} \]

holds.

Problem 2. (2005 Aug/1) Assume that \( (a_n)_{n=1}^{\infty} \) and \( (b_n)_{n=1}^{\infty} \) are sequences of non-negative real numbers such that

(i) \( a_n \leq a_{n+1} \) for any \( n = 1, 2, \ldots \)
(ii) \( b_n \geq b_{n+1} \) for any \( n = 1, 2, \ldots \), and \( \lim_{n \to \infty} b_n = 0. \)
(iii) \( \sum_{n=1}^{\infty} a_n (b_n - b_{n+1}) \) is convergent.

(a) Prove that \( \lim_{n \to \infty} a_n b_n = 0. \)
(b) Show that conclusion (a) may fail if assumption (i) is omitted.

Problem 3. (Part of 2010 Jan/8C) Let \( 0 < \varepsilon < 1 \) be fixed. Let \( a_n = (\varepsilon)^n \). Prove that \( |a_n| \leq \frac{c}{n^{1+\varepsilon}} \) where \( c \) depends only on \( \varepsilon \), not on \( n \).

Problem 4. (2011 Jan/1 - first you can try its easier version: 2010 Aug/3) Let \( K \) be continuous function on the square \([0, 1] \times [0, 1]\), and let \( g \) be a continuous function on \([0, 1]\). Show that there is a unique continuous function \( f \) on \([0, 1]\) so that

\[ f(x) = \int_0^x K(x, y) f(y) dy + g(x). \]

Problem 5. (2011 Jan/3 - similar problems: 2006 Aug/3) Show that there exists a constant \( C \) such that for all \( x \in [0, 2\pi] \) and \( n = 1, 2, \ldots \)

\[ \left| \sum_{k=1}^{n} \frac{\sin(kx)}{k} \right| < C. \]

Hint: Break the sum into two parts for \( kx < 1 \) and \( kx \geq 1 \), respectively.

Date: July 27, 2011.
Problem 6. (2008 Jan/4 - similar problems: 2009 Aug/6, 2006 Jan/1) Let $A$ and $B$ be real numbers. Show that there is a constant $C$ independent of $A$ and $B$, and $N$ so that

$$\left| \int_{-N}^{+N} \left[ e^{i(AX+BX^2)} - 1 \right] \frac{dx}{x} \right| \leq C.$$