Problem 1. (2010 Aug/7R) Let \( \{f_n\}_{n \geq 1} \) be a sequence of continuous functions on the interval \([0, 1]\), and let \( E \) be the set of \( x \in [0, 1] \) for which \( \sup_n |f_n(x)| = \infty \).

Show that \( E \) cannot be \([0, 1] \cap \mathbb{Q}\).

Problem 2. (2010 Jan/8R) Let \( H \) be a Hilbert space over \( \mathbb{R} \), with scalar product denoted by \( \langle \cdot, \cdot \rangle \), and associated norm denoted by \( ||\cdot|| \).

1) Assume that \( (x_n) \) and \( (y_n) \) are sequences in \( H \) such that \( ||x_n|| \leq 1, ||y_n|| \leq 1 \) and \( \langle x_n, y_n \rangle \to 1 \) as \( n \to \infty \). Show that \( (x_n - y_n) \to 0 \) as \( n \to \infty \).

2) Let \( T \) be a continuous linear map from \( H \) into itself.

2.1) Recall what is the definition of the adjoint operator \( T^* \).

2.2) Assume that \( T \) is self adjoint, i.e. that \( T^* = T \). And assume that, for some sequence \( x_n \in H \), with \( ||x_n|| \leq 1 \):

\[
1 = \sup_{||x|| \leq 1} ||T(x)|| = \lim_{n \to \infty} ||T(x_n)||.
\]

Using question 1, show that \( T^2(x_n) - x_n \) tends to 0 as \( n \to \infty \). Conclude that at least one of the 2 operators \( T + 1 \) or \( T - 1 \) is not invertible. Here \( 1 \) denotes the identity map on \( H \) (i.e. \( 1(x) = x \)).

Problem 3. (2010 Aug/6) Let \( I = [0, 1] \), and define for \( f \in L^2(I) \) the Fourier coefficients as

\[
\hat{f}_k = \int_0^1 f(t)e^{-2\pi ikt}dt.
\]

(i) Let \( \mathcal{G} \) be the set of all \( L^2(I) \) functions with the property that \( |\hat{f}_k| \leq |k|^{-3/5} \) for all \( k \in \mathbb{Z} \).

Prove that \( \mathcal{G} \) is a compact subset of \( L^2(I) \).

(ii) Let \( \mathcal{E} \) be the set of all \( L^2(I) \) functions with the property that \( \sum_k |\hat{f}_k|^{5/3} \leq 10^{-10} \). Is \( \mathcal{E} \) a compact subset of \( L^2(I) \)?

Problem 4. (2011 Jan/7R) Is it possible to find a real-valued function \( f \) defined on \([0, 1]\) such that \( \lim_{x \to t} |f(x)| = \infty \) for every rational \( t \in [0, 1] \)?

Problem 5. (2010 Jan/7R) Let \( W \) be the space of continuous functions \( f \) on \([0, 1]\), whose distributional derivative on \((0, 1)\) is an integrable function.

In one variable, this simply means that \( f(x) = f(0) + \int_0^x g(t)dt \), for some integrable function \( g \), and then \( f' = g \). On \( W \) one considers the norm defined by

\[
||f||_W = |f(0)| + \int_0^1 |f'(t)|dt.
\]
Let $\Lambda$ be the space of continuous functions on $[0, 1]$, that are Hölder continuous of order $\frac{1}{2}$ (i.e. functions $f$ such that for some constant $C > 0$, for every $x$ and $y$, $|f(x) - f(y)| \leq C|x - y|^{1/2}$). On $\Lambda$ one considers the norm

$$||f||_\Lambda = |f(0)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{1/2}}.$$ 

Equipped with these norms $W$ and $\Lambda$ are Banach spaces. And it is immediate that the diagonal $\Delta \subset W \times \Lambda$, that is the set of all $(f, f)$ for $f \in W \cap \Lambda$, is a closed subspace of $W \times \Lambda$. You are not asked to justify the above.

(i) Show that if $f \in W$ and $f' \in L^2$ (not only in $L^1$), then $f \in \Lambda$.

(ii) For each integer $N > 0$, set $u_N(x) = \frac{1}{N} \sin(Nx)$. Find constants $A_N$ such that $A_N \to 0$ as $N \to \infty$, and for every $x$ and $y \in \mathbb{R}$

$$|u_N(x) - u_N(y)| \leq A_N|x - y|^{1/2}.$$ 

(iii) Prove that there is a Hölder continuous function $f$, of a Hölder exponent $\frac{1}{2}$, defined on $[0, 1]$ (i.e. $f \in \Lambda$) whose distributional derivative on $(0, 1)$, is not an integrable function (i.e $f \notin W$).

Even if you are not able to prove the result of question (ii), you can use it for applying the open mapping theorem to the projection of $\Delta$ on $\Lambda$, when arguing by contradiction.

**Problem 6.** (2010 Aug/8R) In both parts either construct a distribution in $\mathcal{D}'(\mathbb{R})$ and prove the appropriate estimates, or show that no such distribution exists.

1. Is there a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that for all $\phi \in C_0^\infty(\mathbb{R})$ with compact support in $(0, \infty)$ one has

$$\langle u, \phi \rangle = \int \phi(x)|x|^{-5}dx?$$

2. Is there a distribution $v \in \mathcal{D}'(\mathbb{R})$ so that for all $\phi \in C_0^\infty(\mathbb{R})$ with compact support in $(0, \infty)$ one has

$$\langle v, \phi \rangle = \sum_{n=1}^{\infty} \phi^{(n)}(2^{-n})?$$

**Problem 7.** (2010 Aug/9R) Assume that the sequence $\{x_n\}$ of real numbers is such that $x_n \neq 0$ for some $n$. Take $p \in (1, \infty)$ and let $G$ be the set of all sequences $\{y_n\}$ so that $\{y_n\} \in \ell^p$ and

$$\lim_{N \to \infty} \sum_{n=1}^{N} y_n x_n = 0$$

Prove that $G$ is dense in $\ell^p$ if and only if $\{x_n\} \notin \ell^q$ where $q^{-1} + p^{-1} = 1$. 