Problem 1. (2010 Jan/9C) Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ be the open unit disk, and let $S = \{ w \in \mathbb{C} : |\text{Im} w| < \frac{\pi}{2} \}$ be a horizontal strip centered at zero of width $\pi$.

1) Find an explicit biholomorphic mapping from $S$ to $D$; i.e. find a holomorphic function $\Phi : S \to D$ which is one-to-one and onto and whose inverse is also holomorphic.

2) Let $f : S \to S$ be a holomorphic function with $f(0) = 0$. Show that $\left| e^{f(z)} - 1 \right| \leq \left| e^z - 1 \right|$.

3) Let $f : S \to S$ be a holomorphic function with $f(0) = 0$. What can you conclude about $f$ if $f'(0) = 1$? Why?

Problem 2. (2009 Aug/9C) Let $f$ be a function holomorphic in the open unit disk $D$.

(a) Suppose that $|f(z)| \leq 1$ for every $z \in D$. Prove that $|f^{(N)}(z)| \leq N!(1 - |z|)^{-N}$ for any integer $N \geq 1$ and any $z \in D$.

(b) Suppose that $|f(z)| \leq 1$ for every $z \in D$. Let $0 \neq w_j \in D$, and suppose that $f(w_j) = 0$ for $j = 1, \ldots, N$. Prove that $|f(0)| \leq \prod_{j=1}^{N} |w_j|$. What can you conclude if there is equality?

(c) Let $\{x_n\}$ be a sequence of distinct real numbers such that $|x_n| < \frac{1}{2}$ for every $n \geq 1$, and suppose that $f(x_n)$ is real for every $n \geq 1$. Prove that $f(\overline{z}) = \overline{f(z)}$ for all $z \in D$.

Problem 3. (2010 Aug/7C)

(i) (5pts.) Let $k$ be a positive integer and let $A \geq 0$. Identify explicitly the class of entire holomorphic functions $F : \mathbb{C} \to \mathbb{C}$ with the property

$$\int_0^{2\pi} |F(re^{i\theta})|^2 d\theta \leq Ar^{2k}.$$ 

(ii) (5pts.) Let $f$ be a function holomorphic in the open unit disk $D = \{ z : |z| < 1 \}$ and let $u = \text{Re}(f), v \in \text{Im}(f)$. Suppose furthermore that $u(0) = v(0)$.

Prove that

$$\int_0^{2\pi} u(re^{i\theta})^2 d\theta = \int_0^{2\pi} v(re^{i\theta})^2 d\theta.$$ 

Problem 4. (Part of 2010 Jan/7C) Use contour integration to show that if $x$ is real and $0 < x < 1$ then

$$\int_0^\infty \frac{r^{x-1}}{1+r} dr = \frac{\pi}{\sin(\pi x)}.$$
Problem 5. (2008 Jan/8C) Prove that the improper integrals $\int_0^\infty \sin(x^2)\,dx$ and $\int_0^\infty \cos(x^2)\,dx$ exist, and they both equal $\frac{\sqrt{\pi}}{4}$.

Problem 6. (2010 Aug/9C) Let $a > 0$ and let

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^a}.$$

Show that $f$ is holomorphic in $\mathbb{C}$ and that there are constants $A, B$ so that

$$|f(z)| \leq Ae^{B|z|^1/a}$$

for all $z \in \mathbb{C}$. 