1. (Jan-00.4): Let \( A \in M_n(\mathbb{C}) \) and assume that \( A \) has rank 1.

   (a) What are the possible Jordan canonical forms for \( A \)?

   (b) For each of the forms in (a), find the characteristic and minimal polynomial of \( A \).

   **Solution:**

   a) The rank of an \( k \times k \) Jordan block is \( k \), except when the eigenvalue is zero in which case it is \( k - 1 \) (this follows just by counting pivot columns). Hence the Jordan blocks in the Jordan form can be either (i) \( n - 1 \) of size 1 and eigenvalue 0 and 1 of size 1 and eigenvalue \( \lambda \neq 0 \), or (ii) \( n - 2 \) of size 1 and eigenvalue 0, and 1 of size 2 and eigenvalue 0.

   b) In case (i), the characteristic polynomial is \( x^{n-1}(x-\lambda) \) from the determinant, and the minimal polynomial is \( x(x-\lambda) \) - these two factors are both required, and a trivial check shows that this works. In case (ii), the characteristic polynomial is \( x^n \) from the determinant, and the minimal polynomial is \( x^2 \), since it’s not \( x \), but \( x^2 \) works.

2. (Jan-13.5): Let \( W_n \) be the set of \( n \times n \) complex matrices \( C \) such that the equation \( AB - BA = C \) has a solution in \( n \times n \) matrices \( A \) and \( B \).

   (a) Show that \( W_n \) is closed under scalar multiplication and conjugation.

   (b) Show that the identity matrix is not in \( W_n \).

   (c) Give a complete description of \( W_2 \).

   **Solution:**

   a) If \( C \in W_n \) with \( AB - BA = C \), then \( rC = (rA)B - B(rA) \) and \( P^{-1}CP = (P^{-1}AP)(P^{-1}BP) - (P^{-1}BP)(P^{-1}AP) \).

   b) The observation to make is that \( \text{tr}(AB) = \text{tr}(BA) \), so for a matrix to be in \( W_n \), it must have trace zero. The identity matrix has trace \( n \), which is not zero.

   c) As in part (b), a matrix in \( W_2 \) must have trace zero. From (a) we may assume that \( C \) is in Jordan form, so the possibilities are \( C = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} \) or \( C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \); furthermore it is enough to consider \( C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) by part (a) again. Some calculation shows that \( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), so both are possible. We conclude that \( W_2 \) is the set of \( 2 \times 2 \) trace-zero matrices.
3. (Jan-11.4): Let $V$ be a finite-dimensional $\mathbb{C}$-vector space and $T : V \to V$.

(a) Suppose $W$ is a subspace with $T(W) \subseteq W$. Show that the characteristic polynomial $f_S(x)$ of $S = T|_W$ divides the characteristic polynomial $f_T(x)$ of $T$ on $V$.

(b) Let $\lambda$ be a root of $f_T(x)$ of multiplicity $m$ and $V_{\lambda} = \{ v \in V : T(v) = \lambda v \}$. Show that $1 \leq \dim_{\mathbb{C}} V_{\lambda} \leq m$.

(c) Find $(V, T, \lambda)$ such that $\lambda$ has multiplicity 5 as a root of $f_T(x)$ but $\dim_{\mathbb{C}} V_{\lambda} = 1$.

Solution:

a) Choose a basis for $W$ and extend to one for $V$. Then $T$ is a block-diagonal matrix $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, so $f_T(x) = \det(xI - T) = \det(xI - A) \cdot \det(xI - C) = f_S(x) \cdot \det(xI - C)$.

b) Since $\lambda$ is a root of the characteristic polynomial, the $\lambda$-eigenspace $V_{\lambda}$ is nontrivial so the dimension is at least 1. For the other part, set $W = V_{\lambda}$ and use part (a): the characteristic polynomial $f_S(x)$ divides $f_T(x)$, but on $V_{\lambda}$ the characteristic polynomial only has the root $x = \lambda$ hence must be $(x - \lambda)^{\dim V_{\lambda}}$, which must therefore divide $(x - \lambda)^m$.

b-alt) The elementary way to show this inequality is via the reduced row-echelon form of $\lambda I - T$: the number of rows of all zeroes is the dimension of its kernel hence equal to $\dim_{\mathbb{C}} V_{\lambda}$ (because rank is not changed by row-reduction), but this value is clearly bounded by the number of times 0 appears as a root of the characteristic polynomial of $\lambda I - T$, which is $m$.

b-alt2) Consider the Jordan form of $T$. The dimension of $V_{\lambda}$ is the number of Jordan blocks with eigenvalue $\lambda$, while the multiplicity of $\lambda$ as a root of $f_T(x)$ is the total sum of the sizes of the Jordan blocks with eigenvalue $\lambda$. Clearly, the latter is at least as big as the former.

c) Such a matrix is given by the Jordan block $J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; its kernel is 1-dimensional but its characteristic polynomial is $x^5$. The “alternative” proof makes it clearer where this comes from: the reduced row-echelon form of $\lambda I - T$ is actually forced to be this matrix, as it must be strictly upper-triangular with 4 pivotal columns.

4. (Jan-14.2): Let $F$ be a field and $n$ a positive integer. Let $A \in M_{n \times n}(F)$ such that $A^n = 0$ but $A^{n-1} \neq 0$. Show that any $B \in M_{n \times n}(F)$ that commutes with $A$ is contained in the $F$-linear span of $I, A, A^2, \ldots, A^{n-1}$.

Note: Compare with Aug-94.4.

Solution: The given information says that the minimal polynomial of $A$ is $x^n$, so since $A$ is $n \times n$ we see that its characteristic polynomial is also $x^n$. Now consider the Jordan form of $A$: the only possibility is that it is a single $n \times n$ Jordan block with eigenvalue 0. By the standard property of the rational canonical form, $A$ is conjugate (over $F$) to its Jordan form, and conjugating changes nothing, so we can assume

$$A = \begin{pmatrix} 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ \vdots & \vdots & \cdots & 0 & \cdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$  

It is then straightforward to see that $A^k$ is the matrix whose entries are all 0s, except that the $k$th entries above the diagonal are 1s. To get the final result, one can simply multiply out $AB - BA$ for an arbitrary matrix $B$ to check the claim.

Remark: In fact, this result holds for any $n \times n$ matrix $A$ whose minimal polynomial has degree $n$. Here is a more highbrow proof: $M_{n \times n}(F)$ is a central simple $F$-algebra, and if we take the $n$-dimensional subalgebra $S = F[A]$ generated by $A$, what we want to know is: which elements of $M_{n \times n}(F)$ commute with everything in this subalgebra? Since $M_{n \times n}$ has dimension $n^2$ and $S$ has dimension $n$, the double commutator theorem says that the commutant of $S$ is also $n$-dimensional, hence must actually be just $S$ (since everything in $S$ commutes with everything else in $S$): hence the only matrices which commute with $A$ are those in $S$: namely, matrices that are in the $F$-linear span of $I, A, \cdots, A^{n-1}$. 


5. (Aug-06.5): Let \( A \in M_n(\mathbb{C}) \). Show that the following are equivalent:

(a) The ranks of \( A \) and \( A^2 \) are equal.
(b) The multiplicity of 0 as a root of the minimal polynomial of \( A \) is at most 1.
(c) There is an \( n \times n \) matrix \( X \) such that \( AXA = A, XAX = X, AX = XA \).

**Solution:** We can conjugate without changing anything, so assume without loss of generality that \( A \) is in Jordan form. We show that all three conditions are equivalent to the statement: all Jordan blocks with eigenvalue 0 have size 1.

**a)** The Jordan blocks of eigenvalue not 0 all have full rank in both \( A \) and \( A^2 \) so we need only consider the Jordan blocks with eigenvalue 0. In \( A \) the rank of each such \( k \times k \) block is \( k - 1 \), while in \( A^2 \) the rank is 0. Hence all Jordan blocks with eigenvalue 0 have size 1 if the rank of \( A \) equals the rank of \( A^2 \).

**b)** By basic properties, \( x^2 \) divides the minimal polynomial iff there is a Jordan block of size at least 2 with eigenvalue 0.

**c)** If the Jordan blocks with eigenvalue 0 have size at most 1, say \( A = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \) where \( B \) is invertible (since it has all the nonzero eigenvalues of \( A \) in it); then take \( X = \begin{pmatrix} 0 & 0 \\ 0 & B^{-1} \end{pmatrix} \). Conversely, if such an \( X \) exists, then \( AXA^2 = A \) so \( \text{rank}(A) \leq \text{rank}(A^2) \leq \text{rank}(A) \), so we recover condition (a).

6. (Aug-06.5): Let \( F = \mathbb{F}_q \) and \( M_2(F) \) be the ring of \( 2 \times 2 \) matrices over \( F \).

(a) If \( A \in M_2(F) \) has equal eigenvalues in the algebraic closure of \( F \), show that the eigenvalues of \( A \) belong to \( F \).

(b) Determine the number of nonzero nilpotent matrices in \( M_2(F) \) as a function of \( q \).

**Solution:**

**a)** Let the eigenvalues be \( \lambda, \lambda \). We know that \( \text{Tr}(A) = 2\lambda \) and \( \det(A) = \lambda^2 \), and both of these are in \( F \) (since they are polynomials in the entries of \( A \)). If \( \text{char}(F) \neq 2 \), then \( \lambda = \frac{2\lambda}{2} \in F \). If \( \text{char}(F) = 2 \) then since \( F^\times \) is an abelian group of odd order, squaring is an automorphism of \( F^\times \) so every element of \( F \) has a unique square root in \( F \), so in particular \( \lambda = \sqrt{\det(A)} \in F \).

**b)** A matrix is nilpotent (over any field) iff all its eigenvalues are zero (as either of these is equivalent to its characteristic polynomial being \( x^n \)). For a nonzero \( 2 \times 2 \) matrix the only possibility for its Jordan form is \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). (Alternatively, we could use part (a).) We then want to find the size of the orbit under the conjugation action of \( GL_2(F) \). Since \( |GL_2(F)| = (q^2 - 1)(q^2 - q) \) by the standard vector space argument, by applying the orbit-stabilizer lemma it’s enough to find the invertible matrices which stabilize \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) under conjugation. One can do this just by multiplying out \( \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \) and equating coefficients to see that the matrices which commute with \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) are precisely those of the form \( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \), of which there are \( q(q - 1) \) invertible ones (since the determinant is \( a^2 \)). Whence: there are \( q^3 - 1 \) conjugates of this matrix.

**b-alt)** By the argument given in the Remark to Jan-14.2, since the minimal polynomial of \( M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) has degree 2, the only matrices which commute with it are of the form \( aI + bM \).
7. (Jan-10.4): Let $V$ be finite-dimensional over $F$ and $T : V \to V$, with characteristic polynomial $f(x) \in F[x]$.

(a) Show that $f(x)$ is irreducible in $F[x]$ iff there are no proper nonzero subspaces $W$ of $V$ with $T(W) \subseteq W$.

(b) If $f(x)$ is irreducible and $\text{char}(F) = 0$, show that $T$ is diagonalizable over the algebraic closure $\bar{F}$.

**Solution:**

a) If $f(x)$ is reducible with a factor $g(x)$, then $\ker(g(T))$ is a nontrivial proper subspace which is mapped into itself by $T$. For the other direction, let $W$ be any subspace of $V$ with $T(W) \subseteq W$, and choose a basis to make $T$ block-upper-triangular, say $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where $A$ corresponds to $T|_W$ and $C$ corresponds to $T|_{V/W}$.

Then the characteristic polynomial $p_V(x)$ of $T$ is $\det(xI - T) = \begin{vmatrix} xI - A & -B \\ 0 & xI - C \end{vmatrix} = \det(xI - A) \cdot \det(xI - C)$, which is the product of the characteristic polynomials $p_W(x)$ of $T_W$ and $p_{V/W}(x)$ of $T_{V/W}$. We then see that the characteristic polynomial $T|_W$ divides the characteristic polynomial of $T$.

b) Observe that $f$ has no repeated roots, since otherwise $\text{gcd}(f, f')$ would have positive degree and divide $f$. But then the Jordan form must have all Jordan blocks of size 1, so $T$ is diagonalizable over $\bar{F}$.

8. (Jan-05.4): Let $F$ be an algebraically-closed field and $M_n(F)$ be the ring of $n \times n$ matrices over $F$. Describe those matrices $X \in M_n(F)$ such that all matrices that commute with $X$ are diagonalizable.

**Solution:** The answer is: matrices with distinct eigenvalues in $F$. To see this, first observe that $X$ itself must be diagonalizable since it commutes with itself. Conjugating does not affect the given property, so further assume that $X$ is in Jordan form; hence, diagonal. If $X$ has any equal eigenvalues, then $X$ does not have the given property, since after changing basis to put two equal eigenvalues $\lambda$ in the upper left of the diagonal, $X$ commutes with a Jordan block matrix of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Now if $X$ is diagonal with distinct eigenvalues, we claim that the set of matrices that commute with $X$ are the diagonal matrices. To see this, observe that if $A$ is any matrix and $X$ is the diagonal matrix whose $(i, i)$ entry is $\lambda_i$, then the $(i, j)$ component of $AX -XA$ is $a_{i,j}(\lambda_i - \lambda_j)$, which must be zero. Since the eigenvalues are distinct, we see that all off-diagonal entries of $A$ must be zero, so $A$ is diagonal. Hence, we are done.

**Remark** In fact the result holds over any field, algebraically closed or not. An appropriate usage of the rational canonical form in the above argument will provide a construction of an appropriate non-diagonalizable matrix that commutes with $X$.