1. Are the claims below true or false? Explain your answer.

   (a) There exist two positive definite matrices $A$ and $B$ such that $\det(AB) = 0$.

   (b) If $a, b, c$ are distinct positive numbers, then the matrix $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ is invertible.

   (c) Is the matrix $M$ has the property that all its principal minors are positive, then all its eigenvalues are positive.

   (d) There exist 4 by 4 matrices $A$ and $B$, such that $A$ has rank 2, $B$ has rank 1, and $A + B$ is invertible.

2. Solve the differential equation $\frac{du}{dx} + au = \delta(x)$ by using the Fourier transform.

3. In a model of a laser, two kinds of excited atoms vary according to the equations

   \[
   \frac{dn_1}{dt} = G_1 N n_1 - k_1 n_1
   \]

   \[
   \frac{dn_2}{dt} = G_2 N n_2 - k_2 n_2
   \]

   where $N(t) = N_0 - \alpha_1 n_1 - \alpha_2 n_2$. All the parameters $G_1, G_2, k_1, k_2, \alpha_1, \alpha_2, N_0$ are positive.

   (a) Discuss the stability of the fixed point $n_1 = n_2 = 0$.

   (b) Find and classify any other fixed points that may exist.

   (c) Depending on the values of the parameters, how many qualitatively different phase plane scenarios does the model predict?

4. Poisson problems.

   (a) What is the solution to the following Poisson problem: $\nabla^2 u = \delta(x)$ on $\mathbb{R}^3$, where $\delta(x)$ is the Dirac delta function. No need to justify your solution for this part.

   (b) What is the solution to the following Poisson problem: $\nabla^2 u = C[\delta(x-x_*)-\delta(x+x_*)]$ on $\mathbb{R}^3$, where $C$ is a constant and $x_*$ is a constant vector. Justify your solution.

   (c) Consider part (b) with $x_* = (\epsilon/2, 0, 0)^T$. In the limit $\epsilon \to 0$, how must the distinguished limit of $C = C(\epsilon)$ be chosen in order to give a finite solution $u(x)$ (away from the singular points $\pm x_*$)? Derive the limiting solution $u(x)$ as $\epsilon \to 0$ with your distinguished limit of $C = C(\epsilon)$. 

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5. The “colliding water streams” problem. Consider the shallow water equations:

\[ h_t + (hu)_x = 0, \quad (hu)_t + \left( hu^2 + \frac{1}{2} gh^2 \right)_x = 0 \]  

(1)
on the real line \(-\infty < x < \infty\). These equations can also be written in vector form as

\[ \mathbf{v}_t + (f(\mathbf{v}))(\mathbf{x}) = 0 \]  

(2)
where \( \mathbf{v} = (h, hu)^T \).

(a) What are the Rankine–Hugoniot jump conditions for (2)?

(b) Consider the following initial conditions:

\[ h(x, 0) = h_0 = \text{const}. \quad u(x, 0) = \begin{cases} +u_0 & \text{for } x < 0 \\ -u_0 & \text{for } x > 0 \end{cases} \]  

(3)

Find the solution at general time \( t > 0 \), given the hint below. Determine the form of \( h(x, t) \) and \( u(x, t) \), and sketch some illustrations of your solution.

HINT: The solution at time \( t > 0 \) is constant in each of three regions: to the left of a left-moving shock moving at velocity \( -\sigma \), to the right of a right-moving shock moving at velocity \( +\sigma \), and the middle region between the two shocks. Determine the shock speed \( \sigma \), and determine the values of \( h \) and \( u \) within each of the three regions.

HINT: Without doing any calculations, what must be the value of \( u \) in the middle region between the two shocks? Explain your reasoning.

(c) Is \( h_m \) (the value of \( h \) in the middle region) larger than \( h_0 \) or smaller than \( h_0 \)? Justify your answer in two ways: (i) analytically based on your solution to part (b), and (ii) physically.

6. Multiscale asymptotics. Consider the PDE

\[ u_t = u_{xx} + [1 + \epsilon \gamma]u - \epsilon u^3 \]  

on \([0, \pi]\)

(4)

\[ u(0, t) = 0 \]  

(5)

\[ u(\pi, t) = 0 \]  

(6)

with small parameter \( 0 < \epsilon \ll 1 \). Look for solutions in the form of a two-time-scale asymptotic expansion, with \( T = \epsilon t \):

\[ u(x, t; \epsilon) = u_0(x, t, \epsilon t) + \epsilon u_1(x, t, \epsilon t) + O(\epsilon^2) \]  

(7)

with leading order solution of the special form

\[ u_0(x, t, T) = A(T) \sin x. \]  

(8)

(a) Use the method of multiscale asymptotics to derive an ODE for the evolution of \( A(T) \). Justify the steps of your derivation.

(b) Describe the long-time behavior of \( A(T) \), assuming \( A(0) \) is small, for each possible value of \( \gamma \).