Complete all 6 problems. Each problem is weighted equally.

1. Consider \( u''(x) = f(x) \) on the interval \([0, 1]\) with periodic boundary conditions.

   (a) Derive a 4th-order-accurate finite difference method for this boundary value problem. Show that the local truncation error is \( O(h^4) \) where \( h = \Delta x \) is the mesh width.

   (b) Prove that your method is convergent: (i) Define an appropriate notion of stability, and prove that your method is stable. (ii) Use your consistency and stability results to prove convergence.

2. (a) Consider the ODE system \( du/dt = Au + Bu \) where \( u \in \mathbb{R}^d \), \( A \in \mathbb{R}^{d \times d} \), \( B \in \mathbb{R}^{d \times d} \), \( u(0) = u_0 \), where \( A \) and \( B \) are constant matrices. Show that the following Strang splitting strategy is 2nd-order-accurate: update \( U^n \approx u(n\Delta t) \) to \( U^{n+1} \approx u((n+1)\Delta t) \) using the three-stage procedure

\[
U^* = e^{Ah/2}U^n, \quad U^{**} = e^{Bh}U^*, \quad U^{n+1} = e^{Ah/2}U^{**},
\]

where \( k = \Delta t \).

   (b) Under what conditions on \( A \) and \( B \) is Strang splitting exact? I.e., if \( U^n = u(n\Delta t) \), under what conditions does the strategy from part (a) provide exact equality \( U^{n+1} = u((n+1)\Delta t) \)?

   (c) Consider an extension of the ODE system to the form \( du/dt = Au + Bu + Cu \) where \( C \in \mathbb{R}^{d \times d} \) is also a constant matrix. Derive a 2nd-order-accurate splitting method, where each stage of your method uses at most one of the matrices \( A, B, C \). How many stages does your method use?

3. Consider the advection equation \( u_t + au_x = 0 \) with \( a > 0 \).

   (a) Use a Taylor series expansion of \( u(x,t+k) \) to derive a 2nd-order-accurate finite difference method for the advection equation, with centered spatial derivatives, with time step \( k = \Delta t \) and mesh width \( h = \Delta x \).

   (b) Derive one-sided, 2nd-order-accurate finite difference approximations for \( \partial u/\partial x \) and \( \partial^2 u/\partial x^2 \). (A one-sided approximation of \( \partial u/\partial x \) at grid point \( x_j \) uses the values of \( u \) only at grid points \( x_{j'} \) with \( j' \leq j \).)

   (c) Use a Taylor series expansion of \( u(x,t+k) \) to derive a 2nd-order-accurate finite difference method for the advection equation, with one-sided spatial derivatives.
(d) Use von Neumann stability analysis to derive the stability constraint for the one-sided method from part (c).

4. Prove the following.

(a) If $B \in \mathbb{C}^{n \times n}$, $B = [b_{ij}]$ is strictly diagonally dominant,

$$|b_{ii}| > \sum_{j=1, j \neq i}^{n} |b_{ij}| \quad i = 1, 2, ..., n$$

then $B$ is non-singular.

(b) If $A \in \mathbb{R}^{n \times n}$ is strictly diagonally dominant, then Jacobi iterations to solve $Ax = b$ will converge for any $b \in \mathbb{R}^n$ and any initial guess.

(c) If $A \in \mathbb{R}^{n \times n}$ is strictly diagonally dominant, then Gauss-Seidel iterations to solve $Ax = b$ will converge for any $b \in \mathbb{R}^n$ and any initial guess.

5. Consider the Rayleigh quotient, $r(x) = (x, Ax)/(x, x)$.

(a) Show that for a Hermitian matrix $A \in \mathbb{C}^{N \times N}$ that the Rayleigh quotient gives an eigenvalue estimate whose accuracy is quadratic in the distance between $x$ and the associated eigenvector.

(b) The power method is a slow method for finding the eigenvector corresponding to the largest eigenvalue $\lambda$ of $A$. Starting with $v^{(0)}$ as an initial estimate with $\|v^{(0)}\| = 1$, iteration proceeds as $v^{(k)} = Ap^{(k-1)}/\|p^{(k-1)}\|$, and the $k^{th}$ estimate of the eigenvalue is $\lambda^{(k)} = r(v^{(k)})$. Prove, for $A$ Hermitian and for all $v^{(0)}$ outside a set of measure zero, that

$$\|v^{(k)} - (\pm q_1)\| = O\left(\frac{\lambda_2}{\lambda_1}\right)^{k}, \quad \|\lambda^{(k)} - \lambda_1\| = O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k},$$

where $(\lambda_1, q_1)$ are the largest eigenvalue of $A$ and its associated eigenvector, and $\lambda_2$ is the second largest eigenvalue of $A$.

(c) Why must a method for determining the eigenvalues of a matrix $A \in \mathbb{C}^{N \times N}$, for $N \geq 5$, generally be iterative instead of direct?

6. Let $A \in \mathbb{R}^{N \times N}$ be symmetric positive definite. At each step of a GMRES iteration to solve $Ax = b$ for $b \in \mathbb{R}^N$, the $L_2$ norm of the residual $r_n = b - Ax_n$ is minimized, with $x_n \in \mathcal{K}_n = \text{span}\{b, Ab, A^2b, ..., A^{n-1}b\}$. 

(a) Show that GMRES iteration can be interpreted as the selection of a polynomial $p_n \in P_n$, where $P_n = \{\text{polynomials of degree } \leq n, p(0) = 1\}$, such that $\|p_n(A)b\|$ is minimal, and hence that

$$\frac{\|r_n\|}{\|b\|} \leq \inf_{p_n \in P_n} \|p_n(A)\|.$$  \hspace{1cm} (4)

(b) From (4) we see that the performance of GMRES is controlled by $\inf_{p_n \in P_n} \|p_n(A)\|$. Show that

$$\|p_n(A)\| \leq \kappa(V) \max_i \|p_n(\lambda_i)\|_{\lambda_i \in \Lambda},$$

where $A = V\Lambda V^{-1}$ and $\kappa(V) = \|V\|\|V^{-1}\|$ is the condition number of the matrix $V$.

(c) Consider a symmetric positive definite matrix $A \in \mathbb{R}^{N \times N}$ with $N > 1$ such that $a_{ii} = 3$ and $|a_{ij}| < 2/(N - 1)$. Approximate the convergence rate of GMRES iteration for such a matrix.